

Global existence and smoothness for solutions of viscous Burgers equation. (2) The unbounded case: a characteristic flow study

Jérémy Unterberger^a

^aInstitut Elie Cartan,¹ Université de Lorraine,
B.P. 239, F – 54506 Vandœuvre-lès-Nancy Cedex, France
jeremie.unterberger@univ-lorraine.fr

We show that the homogeneous viscous Burgers equation $(\partial_t - \eta \Delta)u(t, x) + (u \cdot \nabla)u(t, x) = 0$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ ($d \geq 1, \eta > 0$) has a globally defined smooth solution if the initial condition u_0 is a smooth function growing like $o(|x|)$ at infinity. The proof relies mostly on estimates of the random characteristic flow defined by a Feynman-Kac representation of the solution. Viscosity independent a priori bounds for the solution are derived from these. The regularity of the solution is then proved for fixed $\eta > 0$ using Schauder estimates.

The result extends with few modifications to initial conditions growing abnormally large in regions with small relative volume, separated by well-behaved bulk regions, provided these are stable under the characteristic flow with high probability. We provide a large family of examples for which this loose criterion may be verified by hand.

Keywords: viscous Burgers equation, conservation laws, maximum principle, Schauder estimates, Feynman-Kac formula, characteristic flow.

Mathematics Subject Classification (2010): 35A01, 35B45, 35B50, 35K15, 35Q30, 35Q35, 35L65, 76N10.

¹Laboratoire associé au CNRS UMR 7502

Contents

1	Introduction and summary of results	2
1.1	Introduction	2
1.2	Summary of results	5
2	A prototypical example	8
2.1	Generalities	8
2.2	The non-viscous case	10
2.3	The viscous case	13
3	More general initial data	16
3.1	Non-viscous case	17
3.2	Viscous case	20
4	Proof of the convergence of the scheme	24
4.1	Scheme of proof	25
4.2	Gradient bounds	26
4.2.1	Gradient bounds in the initial regime	26
4.2.2	Large-time bounds for the gradient	31
4.2.3	Bounds for $\nabla^2 u^{(m)}$	33
4.3	Bounds for $v^{(m)}$	34
4.4	Gradient bounds for $v^{(m)}$	34
5	Appendix	36

1 Introduction and summary of results

1.1 Introduction

The $(1 + d)$ -dimensional viscous Burgers equation is the following non-linear PDE,

$$(\partial_t - \eta \Delta + u \cdot \nabla)u = 0, \quad u|_{t=0} = u_0 \quad (1.1)$$

for a velocity $u = u(t, x) \in \mathbb{R}^d$ ($d \geq 1$), $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where $\eta > 0$ is a viscosity coefficient, Δ the standard Laplacian on \mathbb{R}^d , $u \cdot \nabla u = \sum_{i=1}^d u_i \partial_{x_i} u$ the convection term, and g a continuous forcing term. Among other things, this fluid equation describes the hydrodynamical limit of interacting particle systems [12, 8], is a simplified version without pressure of the incompressible Navier-Stokes equation, and also (adding a random forcing term in the right-hand side) an interesting toy model for the study of turbulence [1].

The traditional strategy to show a priori estimates for this equation, see e.g. [9], is to combine integral L^2 -estimates (the simplest of which coming from the energy balance equation) with the maximum principle. The latter, valid for any transport equation – but not for the related Navier-Stokes equation – implies a uniform bound for the supremum $\|u_t\|_\infty$ of the solution, $\|u_t\|_\infty \leq \|u_0\|_\infty$.

In a previous article [16], we showed that the maximum principle alone was enough to show global existence and boundedness of the solution, provided the initial solution is *bounded* together with its derivatives to order 2. In particular, it is not necessary to assume that u_0 or g are in L^2 -spaces

to solve the equation. Also, our bounds do not grow exponentially in time, contrary to the classical bounds based on energy estimates, see e.g. [9].

In the present work, we aim at *relaxing the boundedness hypothesis as much as possible*. If the initial condition is unbounded, then the maximum principle does not make sense any more. For solutions of some *scalar* parabolic equations, e.g. of viscous Hamilton-Jacobi equations, the comparison principle allows one to define viscosity solutions growing at infinity [3]. However, here u is not scalar, nor can it be reduced in general to the solution of a Hamilton-Jacobi equation (save in dimension 1), so it is not at all clear if such a strategy can work. Instead we tackle the problem *from a dynamical system perspective* and ask ourselves: *can one find general criteria ensuring that characteristics of the flow do not blow up ?*

It turns out that this question is really the crux of the problem. Let us explain roughly why in the case of zero viscosity ($\eta = 0$). Recasting this Eulerian fluid equation into a Lagrangian language, u is constant along its (time-reversed) characteristics, defined as the solutions of the ordinary differential equations $\frac{d}{ds}x(t; s, x) = u(t - s, x(t; s, x))$ with initial condition x ; in other words, $u(t, x) = u(t - s, x(t; s, x))$. In particular $u(t, x) = u_0(x(t; t, x))$ is a priori well defined if u_0 is, no matter how large u_0 can be. The argument is clearly faulty as the characteristic $x(t; s, x)$ may indeed blow up if u_0 grows too fast at infinity. This is clear if one replaces u by the approximation \tilde{u} (denoted $u^{(1)}$ later on) defined by: $\tilde{u}(t, x) := u_0(\tilde{x}(t; t, x))$, $\tilde{x}(t; \cdot, x)$ solving the above differential equation, but with the velocity $u(t - s, \cdot)$ approximated by the initial velocity $u_0(\cdot)$, namely, $\frac{d}{ds}\tilde{x}(t; s, x) = u_0(\tilde{x}(t; s, x))$. This equation does not blow up in finite time if u_0 is Lipschitz and has sublinear growth at infinity. Since linear growth is really a border case, we shall rather consider as *prototypical initial velocity* a function with *strictly sublinear* growth, namely, $|u_0(x)| = O_{|x| \rightarrow \infty}(|x|^{1/\kappa})$, $\kappa > 1$, for which $\tilde{x}(t; t, x)$ grows for large time like $t^{\kappa/(\kappa-1)}$. But then one may go one step further and remark that the *instantaneous value* of u_0 at some point is not so important. Indeed, in one dimension, the non blow-up criterium states that the time needed to go from x to x' (equal to $\int_x^{x'} \frac{dy}{u_0(y)}$ if e.g. $x < x'$ and $u_0 > 0$) must diverge when $|x'| \rightarrow \infty$; this does not prevent u_0 from becoming arbitrary large in regions with small relative size, provided these are separated by large bulk intervals where u_0 grows sublinearly and which therefore take up a large time to cross. In short, we are happy if $\tilde{x}(t; t, x) - x = O(t^{\kappa/(\kappa-1)})$ for t large.

Surely enough, this last criterion should not be taken seriously for a number of obvious reasons (it is dimension-dependent, what t large means is not clear, the connection to the original non-linear equation is not clear, what happens in case of non-zero viscosity, etc.), but it really is the inspiration of the present work. Let us sketch the answer to some of the objections we have just raised. First, as in [16], we use the following scheme of successive approximations to the solution. We solve inductively the linear transport equations,

$$u^{(-1)} := 0; \tag{1.2}$$

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla)u^{(m)} = 0, \quad u^{(m)}|_{t=0} = u_0 \quad (m \geq 0). \tag{1.3}$$

If the sequence $(u^{(m)})_m$ converges locally in $C^{1,2}$ -norms, then the limit is a fixed point of (1.3), hence solves the Burgers equation. The Feynman-Kac formula implies the following well-known representation of the solution of (1.3) in terms of *random characteristics* $X^{(m)}(t, \cdot)$,

$$u^{(m)}(t, x) = \mathbb{E}[u_0(X^{(m)}(t, x))], \tag{1.4}$$

where $X^{(m)}(t, x) := X^m(t; t, x)$ is the solution at time t of a stochastic differential equation driven by

a standard Brownian motion B ,

$$dX^{(m)}(t; s, x) = u^{(m-1)}(t-s, X^{(m)}(t; s, x))ds + dB_s, \quad (1.5)$$

started at $X^{(m)}(t; 0, x) := x$.

In section 2 we concentrate on *prototypical initial velocities*, i.e. study Burgers equation under the hypothesis

$$(\text{Hyp1}) \quad |u_0(x)| \leq U(1 + |x|)^{1/\kappa}, \quad x \in \mathbb{R}^d \quad (1.6)$$

with $\kappa > 1$ and $U \geq 1$. Solving for the random characteristic $X^{(1)}$ (which coincides with the above deterministic characteristics \tilde{x} in the zero viscosity case), we prove that for t large, with high probability,

$$|X^{(1)}(t; s, x) - x| = O\left(\max\left((Ut)^{\kappa/(\kappa-1)}, Ut|x|^{1/\kappa}\right)\right), \quad (1.7)$$

thus retrieving for t large the behaviour in $O(t^{\kappa/(\kappa-1)})$. Then we note that $X^{(m)}, m \geq 2$ solves essentially the same equation as $X^{(1)}$ since $u^{(m-1)}(t-s, y) = \mathbb{E}\left[u_0(X^{(m-1)}(t-s, y))\right]$ is the average of u_0 on some weighted cloud of points in a neighbourhood of y . At this point it is natural to introduce what we call a *generalized flow with initial velocity* u_0 (see Definition 2.6). Roughly speaking, at least in the non-viscous case, this is an ordinary differential equation of the form $\frac{d}{ds}y(t; s, x) = u_0(X(t; s, y(t; s, x)))$ where $X(t; s, \cdot)$ satisfies an estimate of the same form as $X^{(1)}(t; s, \cdot)$ (see eq. (1.7)). In the *viscous case*, we first convert the *stochastic* differential equation (1.5) into an *ordinary* differential equation with random coefficients by subtracting the additive noise B (see section 2.3). Then *viscous generalized flows* (see Definition 2.8) are (non-viscous) generalized flows, in which spatial arguments have been translated by the noise. Now the interesting property about generalized flows $y(t; \cdot, x)$ is that they themselves satisfy some version of (1.7), where U is the constant appearing in (Hyp1) (see Lemmas 2.7, 2.9). As a result, we are able to obtain inductively bounds for $X^{(m)}$ of the type (1.7) which are *uniform in m* .

At this point, one would be tempted to define an *admissible* initial velocity as a function u_0 for which the inductive Lemmas 2.7, 2.9 hold. As pointed out above, the restriction 'for t large' is *essential*: should we require that (1.7) hold for t small, this would directly imply a sublinear bound on the velocity. Actually, working out the computations, it appears very soon that $t \gtrsim U^{-1}$ is the right condition. Now, while for a given function u_0 the conclusions of Lemmas 2.7, 2.9 may be eventually verified by hand, it turns out that, leaving aside the settled case of functions satisfying (Hyp1), it is difficult to produce any interesting example of admissible velocity. The reason is of topological origin: we need some criterion ensuring inductively the *stability* under the characteristic flows of the *safe zones* where u_0 is sublinear. To be more specific (see section 3), we *assume* that u_0 is sublinear in some 'bulk' *safe* region \mathcal{S} (connected or not), while it is essentially arbitrary in a countable disjoint union of 'thin' *dangerous* regions $(\mathcal{A}_i)_{i \in I}$. In Definition 3.1 we choose these to be annuli, but clearly this is only a reasonable, practical choice. The important thing is that, sticking to the non-viscous case for the time being, *provided* the safe zones are 'fat' enough, one is able to prove inductively a *safe zone stability property* stating that

$$\left(x^{(m-1)}(t; s, x) \in \mathcal{S}(t-s), t \geq s \geq 0\right) \implies \left(x^{(m)}(t; s, x) \in \mathcal{S}(t-s), t \geq s \geq 0\right),$$

where $t \mapsto \mathcal{S}(t)$ is some decreasing family of non-empty subsets with $\mathcal{S}(0) = \mathcal{S}$ (see Theorem 3.1). In this way we show that $x^{(m)}(t; s, x) \in \mathcal{S}(t-s)$ for all m as soon as $x \in \mathcal{S}(t)$. Let $\mathcal{A}(t) := \mathbb{R}^d \setminus \mathcal{S}(t)$ be the *enlarged dangerous zone*. If $x \in \mathcal{A}(t)$, then x may a priori jump to the boundary of $\mathcal{A}(t)$ in arbitrarily short time, after which it cannot escape from the safe zone any more due to the safe zone

stability property. If $\mathcal{A}(t)$ is still a disjoint union $(\mathcal{A}_i(t))_{i \in I}$ of thin regions, then this may (and does under our assumptions for $(\mathcal{A}_i)_{i \in I}$) prove enough to show a uniform bound of the type (1.7). Thus the safe zone stability property is an efficient replacement for the inductive property of Lemmas 2.7, 2.9.

A straightforward generalization of these arguments to the *viscous case* appears to be impossible at first sight, since one may always fall into the dangerous zone by translating by some random amount the spatial arguments. Even though these random amounts are bounded *in average*, without additional assumptions on u_0 , it may happen, with a small but nonzero probability, that random characteristics blow up. So much for the debit side. On the credit side, one sees that the translation by random paths $(B_t)_{t \geq 0}$ bounded by $o(t)$ for t large (which is the case of the overwhelming majority of random paths since B_t is roughly of order \sqrt{t}) should not affect the usual displacement bound in $O\left(\max\left((Ut)^{\kappa/(\kappa-1)}, Ut|x|^{1/\kappa}\right)\right)$ (see eq. (1.7)) since $o(t) \ll (Ut)^{\kappa/(\kappa-1)}$ for $t \geq U^{-1}$ (see (2.44) for a more quantitative statement). In short, as emphasized in section 2.3, *convection prevails over diffusion* in normal conditions. Since the opposite regime where diffusion prevails over convection is highly improbable, only very mild assumptions (e.g. polynomial growth at infinity) are $u_0|_{\mathcal{A}}$ is required to extend the safe zone stability property argument to the viscous case. A precise statement may be found e.g. in Lemma 3.4.

Once one has a uniform control of the random characteristics, and some polynomial 'a priori' bound on u_0 , one may start about proving the convergence of the scheme (1.3), which is the subject of section 4. From that point on, we follow a more conventional course of action, which is sketched in the next paragraph.

1.2 Summary of results

The general assumptions on the initial velocity u_0 are written down in the preamble of section 4. Fix $U \geq 1, \kappa > 1$. We demand the following: (i) u_0 is C^2 ; (ii) $u_0, \nabla u_0$ and $\nabla^2 u_0$ grow at most polynomially at infinity (these we call *a priori bounds* for u_0 , see (4.1)); plus a third condition (iii) stating roughly that the characteristic flows $s \mapsto X^{(m)}(t; s, x)$ may be estimated for $t \gtrsim U^{-1}$ like the deterministic flow $s \mapsto y(s, x)$ defined by the ordinary differential equation $\frac{d}{ds}y(s, x) = (1 + |y(s, x)|)^{1/\kappa}$ with initial condition $y(0, x) = x$, except when $\sup_{0 \leq s \leq t} |B_s|$ overrides the usual displacement bound (1.7), the latter condition defining the so-called highly improbable *abnormal regime* where diffusion prevails over convection. Depending on whether one wants *examples* built following the above arguments (with explicit 'safe' and 'dangerous' zones, etc.) which are *sufficient* to ensure such estimates, or one rather looks for more or less 'necessary' conditions *a minima* on the characteristics in the abnormal regime ensuring that all subsequent estimates (on $u^{(m)}, \nabla u^{(m)} \dots$) remain unaffected, one obtains different versions of (iii). The *sufficient* condition (iii) is based on Definition 3.1:

Theorem 1 (see Definition 3.1, Theorem 3.2 and (3.44)) *Let $(R_n)_{n \geq 1}$ be an increasing sequence, $1 \leq R_1 < R_2 < R_3 < \dots$ such that, for all $i \geq 1$,*

$$R_{2i} - R_{2i-1} \leq R_{2i-1}^{1/\kappa}, \quad (1.8)$$

$$R_{2i+1} \geq 4R_{2i}. \quad (1.9)$$

Let $\tilde{u}_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an initial velocity satisfying (Hyp1) (see (1.6)) for some constants $U \geq 1, \kappa > 1$. Let $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be any Lipschitz function coinciding with \tilde{u}_0 outside the union of annular

'dangerous zones' $\cup_{i \geq 1} \mathcal{A}_i$, $\mathcal{A}_i := B(0, R_{2i}) \setminus B(0, R_{2i-1})$, and satisfying the a priori bounds (4.1). Let also $M_t := 1 + \frac{\sup_{0 \leq s \leq t} |B_s|}{\sqrt{t}}$. Then the sequence of noise-translated characteristics $(Y^{(m)}(t; \cdot, x))_{m \geq 0}$, $Y^{(m)}(t; s, x) := X^{(m)}(t; s, x) - B_s$, satisfies the following uniform in m estimates:

$$|Y^{(m)}(t; s, x) - x| \lesssim \langle Ut \rangle \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, |x|)^{1/\kappa} \quad \text{if } M_t \sqrt{t} \leq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa}); \quad (1.10)$$

in the normal regime, otherwise

$$|Y^{(m)}(t; s, x) - x| \lesssim \left(\frac{M_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa \quad (1.11)$$

Furthermore, estimates (1.10), (1.11) imply for $u^{(m)}$, $m \geq 0$ defined by Feynman-Kac's formula (1.4)

$$|u^{(m)}(t, x)| \lesssim K_0(|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\frac{q}{2} + \frac{1}{\kappa}}. \quad (1.12)$$

On the other hand, bounds (1.11) in the *abnormal regime* $M_t \sqrt{t} \geq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$ may be considerably softened without harming ulterior bounds. In particular, substituting to (1.11) the condition

$$|Y^{(m)}(t; s, x) - x| \lesssim (M_t \sqrt{t})^{\kappa'} \quad (1.13)$$

for some arbitrary exponent $\kappa' \geq 1$, one still has (1.12). Demanding only (1.10) and (1.13), we get our 'necessary' condition (iii'). Of course, it remains to be proved that there are different choices of dangerous zones – or, from a wider perspective, of functions u_0 – for which (1.13) holds but not (1.11). In any case, bounds in section 4 are based on (1.13).

Let us comment on conditions (1.8), (1.9). Condition (1.8) states that the *width of the dangerous zone* \mathcal{A}_i is *smaller* than the expected displacement $0 \left(\max \left((Ut)^{\kappa/(\kappa-1)}, Ut|x|^{1/\kappa} \right) \right)$ (see (1.7)) for all $t \geq U^{-1}$. Condition (1.9) states that the *width of the safe zone* $B(0, R_{2i+1}) \setminus B(0, R_{2i})$ is *larger* than the expected displacement for $|x| \gg \langle Ut \rangle^{\kappa/(\kappa-1)}$. The latter condition (characteristic of the so-called *short-time regime*, where $\max \left((Ut)^{\kappa/(\kappa-1)}, Ut|x|^{1/\kappa} \right) \lesssim |x|$) comes up naturally right from the beginning (see section 2.1). There is nothing special about the coefficient 4 in (1.9), and our results carry through if $R_{2i} - R_{2i-1} \leq CR_{2i-1}^{1/\kappa}$, $R_{2i+1} \geq (1 + \varepsilon)R_{2i}$ with $C, \varepsilon > 0$ arbitrary, but then implicit constants also depend on C, ε , instead of depending only on the dimension d and on the exponents κ, κ' .

From a logical point of view, the above Theorem is inaccurate since it provides a priori bounds for objects such as $Y^{(m)}(\cdot; \cdot, \cdot)$, $u^{(m)}(\cdot, \cdot)$ without proving their existence. In particular, one must prove inductively that $(u^{(m)})_{m \geq 0}$ are C^1 , so that the transport equations (1.3) are well-posed and we can use Cauchy-Lipschitz's theorem to define uniquely the characteristics. Ultimately we prove the following:

Theorem 2 (see sections 4.2, 4.3, 4.4) *Assume that hypotheses (1.10), and (1.11) (or more generally (1.13)) hold, and that $u_0, \nabla u_0, \nabla^2 u_0$ satisfy the following a priori bounds (see (4.1)),*

$$|u_0(x)| \leq K_0(1 + |x|)^{\frac{q}{2} + \frac{1}{\kappa}}, \quad |\nabla u_0(x)| \leq K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}, \quad |\nabla^2 u_0(x)| \leq K_2(1 + |x|)^{\frac{3}{2}(\frac{q}{2} + \frac{1}{\kappa})} \quad (1.14)$$

with

$$K_0 \leq U^{\frac{\beta}{2}+1}, \quad K_0 \leq K_1^{1/2}, \quad U \leq K_1 \leq K_2^{2/3}, \quad (1.15)$$

for some exponents $\alpha, \beta \geq 0$.

Let $v^{(0)} := u^{(0)}$ and $v^{(m)} := u^{(m)} - u^{(m-1)}$ ($m \geq 1$). Fix $\gamma \in (0, 1)$. Then there exists a universal constant $C = C(d, \kappa, \kappa', \alpha, \beta, \gamma) > 1$ such that, for $m \geq 0$,

$$|\nabla u^{(m)}(t, x)| \leq C^2 K_1 (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}}; \quad (1.16)$$

$$|\nabla^2 u^{(m)}(t, x)| \leq C^4 K_2 (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{3(\frac{\alpha}{2} + \frac{1}{\kappa})}; \quad (1.17)$$

$$|v^{(m)}(t, x)| \leq C K_0 (t/m T_{\min}(t, x))^m (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}}; \quad (1.18)$$

$$|\nabla v^{(m)}(t, x)| \leq C^3 K_2^{2/3} (t/m \tilde{T}_{\min}(t, x))^{\gamma m/2} (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}} \quad (1.19)$$

where

$$T_{\min}(t, x) := \left(C^3 K_1 (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}} \right)^{-1}, \quad \tilde{T}_{\min}(t, x) := \left(C^3 K_2^{2/3} (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}} \right)^{-1}. \quad (1.20)$$

Estimates (1.18, 1.19) imply convergence in absolute value of the series $\sum_{m \geq 0} v^{(m)}$, $\sum_{m \geq 0} \nabla v^{(m)}$, from which it may be concluded by standard arguments that the limit v satisfies Burgers' equation. Theorems 1 and 2 must actually be proved simultaneously since they are based on induction (the a priori bounds at rank $m - 1$, m proved in Theorem 1 are used to prove rank m gradient estimates (1.16) of Theorem 2, from which one can justify the a priori bounds at rank $m + 1$, etc.).

Let us comment on a priori bounds (1.14), and in particular on (1.15). As noted in our previous article [16], dimensional analysis, confirmed by the initial perturbative expansion but also by Schauder estimates for large t , tells us that $u, \nabla u, \nabla^2$ should scale like L^{-1}, L^{-2}, L^{-3} for some reference length L depending on the initial condition, at least for bounded solutions. (In our setting where u_0 may increase polynomially, we have included an extra reference length ≈ 1 .) This account for the relations between the exponents appearing in (1.14), (1.15)), except for β which is arbitrary. Note that β does not appear in the bounds (1.16, 1.17, 1.18, 1.19), except in the numerical constant C . Finally the hypotheses $K_0 \leq K_1^{1/2}, U \leq K_1 \leq K_2^{2/3}$ may be discarded provided one defines as in [16] some constant $K := \max(U, K_0^2, K_1, K_2^{2/3})$ homogeneous to an inverse length, and replaces K_0, K_1, K_2 in (1.16, 1.17, 1.18, 1.19) by $K^{1/2}, K, K^{3/2}$, thus equating \tilde{T}_{\min} with T_{\min} .

Let us finally say some words about the strategy of proof (see section 4.1 for more details), which follows closely that of our previous article [16]. In principle, we would like to prove the gradient bounds (1.16), (1.17), (1.19) by using Feynman-Kac's formula and hypotheses (4.1), (1.10), (1.13) in an initial regime $t \leq T_{\min}(0, x) = (C^3 K_1 (1 + |x|)^{\alpha + \frac{2}{\kappa}})^{-1}$, beyond which exponential factors due to separation of trajectories become large. However this makes no sense in itself since $T_{\min}(0, x) \rightarrow_{|x| \rightarrow \infty} 0$. Furthermore, we are not even able to prove such estimates if one takes into account the contribution of the 'abnormal regime' to the expectation appearing in Feynman-Kac's formula. The solution to these problems is to rewrite $u^{(m)}$ as the sum of a series with general term $\underline{u}^{(m,n)} := u^{(m,n)} - u^{(m,n-1)}$, where $u^{(m,n)}$, $n \geq 0$ solves a *penalized* transport equation meant as a smoothened substitute of the original equation solved on the dyadic ball $B(0, 2^n)$ (see section 4.2). Then $\nabla u^{(m,n)}$, and similarly $\nabla^2 u^{(m,n)}, \nabla v^{(m)}$ may be proved inductively to satisfy (1.16, 1.17, 1.19) for $t \leq T_n := (C^3 K_1 (2^n)^{\alpha + \frac{2}{\kappa}})^{-1} \approx T_{\min}(0, 2^n)$. Furthermore, for x small, namely, if $|x| \ll 2^n$,

then Gaussian bounds for Brownian motion imply that $\nabla u^{(m,n)}(t, x)$, $\nabla^2 u^{(m,n)}(t, x)$, $\nabla v^{(m)}$ are exponentially small; intuitively this is clear since the only contribution to $\nabla \underline{u}^{(m,n)}$ comes from characteristics $X^{(m)}(t; \cdot, x)$ which go very far away from x , crossing the boundary of $B(0, 2^n)$. Extension of these bounds to larger t is proved using home-made (interior) Schauder estimates proved in our previous article [16].

Finally, the series in n converge thanks to the estimates in the small x regime.

Notations: we let $\langle t \rangle := \max(1, t)$ for $t \in \mathbb{R}_+$, $\langle x \rangle := \max(1, |x|)$ for $x \in \mathbb{R}^d$. Also, given two functions f, g , $f \lesssim g$ (resp. $f \gtrsim g$) means: there exists an overall constant C (depending only on d and on the exponents $\kappa, \kappa', \alpha, \beta, \gamma$ possibly) such that $|f(x)| \leq C|g(x)|$ (resp. $|f(x)| \geq C|g(x)|$) on the set where f, g are defined. Then $f \approx g$ means: $f \lesssim g$ and $f \gtrsim g$.

2 A prototypical example

In this section we are only interested in providing a priori bounds for the random paths $X^{(m)}(\cdot; \cdot, \cdot)$, assuming that the sequence of transport equations (1.3) admits a unique smooth solution represented by Feynman-Kac's formula (1.4, 1.5). By rescaling we assume $\eta = 1$ (*viscous case*) or $\eta = 0$ (*non-viscous case*), the latter case serving essentially as an illustration.

We assume throughout that u_0 is C^1 ; this is a priori not absolutely necessary (because of the regularizing properties of the heat kernel), but reasonable if one wants to define properly the random characteristics down to time 0. We make here the following hypothesis:

(Hyp1) There exist constants $U \geq 1, \kappa > 1$ such that $|u_0(x)| \leq U(1 + |x|)^{1/\kappa}$.

The condition $U \geq 1$ is of course inessential; it avoids having to distinguish between the factors $O(U)$ and the factors $O(1 + U)$ which pop up in the proofs. Assuming u_0 is small, optimal results using our arguments may be obtained by rescaling the solution and the time-variable in such a way that $\sup_{x \in \mathbb{R}^d} \frac{|u_0(x)|}{(1+|x|)^{1/\kappa}} = 1$, but mind that this reintroduces a viscosity parameter into the story, producing in turn a time rescaling in the bounds (which is very easy to write down by following the computations step by step).

A prototypical family of natural examples is of course smooth functions u_0 satisfying $u_0(x) = F(\frac{x}{|x|})U|x|^{1/\kappa}$ outside $B(0, 1) := \{x \in \mathbb{R}^d \mid |x| < 1\}$, where $F : S^d \rightarrow S^d$ is a smooth function preserving the sphere $S^d := \{|x| = 1\}$.

In section 3 we shall see that a priori bounds similar to those shown in this section may be obtained for much more general initial data.

2.1 Generalities

We study in this paragraph the flows of ordinary differential equations (*ode's* for short) of the type $\dot{x} = u_0(x)$ where u_0 satisfies (Hyp1) with parameters U, κ such that $U \geq 1, \kappa > 1$.

We start by introducing a family of typical ode's depending on a parameter $x_{\min} \geq 0$ which we call *cut-off*.

Definition 2.1 ($x_{\min} > 0$) Let $\Phi_{\kappa, U, x_{\min}}(t, x)$ be the solution at time $t \geq 0$ of the scalar ode $\frac{d}{dt}x(t) = U(x_{\min} + |x(t)|)^{1/\kappa}$ started at $x(0) = x \in \mathbb{R}$.

Solving for $x \geq 0$, one gets ($\kappa > 1$ is of course necessary to get a global solution)

$$x(t) = \left((x + x_{\min})^{\frac{\kappa-1}{\kappa}} + \frac{\kappa-1}{\kappa} U t \right)^{\frac{\kappa}{\kappa-1}} - x_{\min}, \quad t, x \geq 0. \quad (2.1)$$

The above solution extends to $t \leq 0$ or $x \leq 0$ as follows. If $x \leq 0$, $\Phi_{\kappa,U,x_{\min}}(t, x)$ reaches 0 after a time $t = T_{\kappa,U}(x) = U^{-1} \frac{\kappa}{\kappa-1} \left((x_{\min} + |x|)^{(\kappa-1)/\kappa} - x_{\min}^{(\kappa-1)/\kappa} \right)$, after which we define $\Phi_{\kappa,U}(t, x) := \Phi_{\kappa,U}(t - T_{\kappa,U}(x), 0) > 0$. Then (by symmetry) $\Phi_{\kappa,U}(-t, -x) = -\Phi_{\kappa,U}(t, x)$.

By convention, we let $\Phi_{\kappa,U}(t, x) = \lim_{x_{\min} \rightarrow 0^+} \Phi_{\kappa,U,x_{\min}}(t, x)$.

The ode's we are interested in are ode's on \mathbb{R}^d . Fix $U \geq 1$ and $\kappa > 1$.

Definition 2.2 An ode $\frac{d}{ds}x(s) = v(s, x(s))$ in \mathbb{R}^d has velocity bounded by $U(x_{\min} + |\cdot|)^{1/\kappa}$ on $[0, t]$ if $|v(s, y)| \leq U(x_{\min} + |y|)^{1/\kappa}$ for all $s \in [0, t]$ and $y \in \mathbb{R}^d$.

If the velocity field v satisfies this property, we write $v \in \mathcal{V}_{\kappa,U,x_{\min}}(t)$.

Definition 2.3 Let $B_{\kappa,U,x_{\min}}(t, x) := \cup_{v \in \mathcal{V}_{\kappa,U,x_{\min}}(t)} \{(x(s))_{0 \leq s \leq t} \mid (x(s))_{0 \leq s \leq t} \text{ solution of the ode } \frac{d}{ds}x(s) = v(s, x(s)) \text{ started at } x(0) = x\}$. Let also

$$B_{\kappa,U}(t, x) := \cup \{B_{\kappa,U,x_{\min}}(t, x); x_{\min} \leq 1\}. \quad (2.2)$$

Let us first study $B_{\kappa,U}(t, x)$. If the ode $\frac{d}{ds}x(s) = v(s, x(s))$ started at x has a velocity bounded by $U(1 + |\cdot|)^{1/\kappa}$, then $\frac{d}{ds}|x(s)| \in [-U(1 + |x(s)|)^{1/\kappa}, U(1 + |x(s)|)^{1/\kappa}]$. Thus $B_{\kappa,U}(t, x) \subset B(x, R_t(|x|))$, where $R_t(|x|) = \max(x_+(t) - |x|, |x| - x_-(t))$ and $x_{\pm}(t)$ are the solution at time t of the scalar ode's $\frac{d}{ds}x_+(s) = U(1 + |x_+(s)|)^{1/\kappa}$, resp. $\frac{d}{ds}x_-(s) = -U(1 + |x_-(s)|)^{1/\kappa}$ started at $|x|$.

The reader may easily check by solving either of these ode's and comparing to (2.1) that $R_t(|x|) \approx \max(\Phi_{\kappa,U}(t, |x|) - |x|, |x| - \Phi_{\kappa,U}(-t, |x|))$ as soon as $|x| \gtrsim 1$ or $Ut \gtrsim 1$. Then clearly $|x| - \Phi_{\kappa,U}(-t, |x|) \leq \Phi_{\kappa,U}(t, |x|) - |x|$. In absolute generality, it holds $R_t(|x|) \lesssim \Phi_{\kappa,U}(\max(t, U^{-1}), |x|) - |x|$; the short-time regime $t \lesssim U^{-1}$ is rather uninteresting and need not be discussed in greater details. Looking more closely at the solution $x(t)$ of (2.1) with $x_{\min} \leq 1$, we see that there are two regimes, the *long-time regime* where $|x| \ll |Ut|^{\kappa/(\kappa-1)}$ and

$$|x| \ll |Ut| |x|^{1/\kappa} \ll |x(t) - x| \approx |x(t)| \approx |Ut|^{\kappa/(\kappa-1)}, \quad (2.3)$$

and the opposite *short-time regime*, $|x| \gg |Ut|^{\kappa/(\kappa-1)}$, where

$$|Ut|^{\kappa/(\kappa-1)} \ll |x(t) - x| \approx |Ut| |x|^{1/\kappa} \ll |x| \quad (2.4)$$

is small. Note that

$$|x(t) - x| \lesssim \max(|Ut|^{\kappa/(\kappa-1)}, |Ut| |x|^{1/\kappa}) \quad (2.5)$$

for all values of t and x .

All these estimates generalize straightforwardly to *small cut-offs*, $x_{\min} \lesssim (Ut)^{\kappa/(\kappa-1)}$: namely, for such values of x_{\min} , $x(s) \in B_{\kappa,O(U)}(t, x)$ for $s \in [0, t]$, as easily shown from the previous computations.

Things get different when x_{\min} is *large*, say, $x_{\min} > (Ut)^{\kappa/(\kappa-1)}$. Taylor expanding (2.1) started from $x > 0$, one sees that, for all $t > 0$,

$$x(t) = (x + x_{\min})(1 + O(Ut x_{\min}^{-(\kappa-1)/\kappa})) - x_{\min} = x + O(Ut x_{\min}^{1/\kappa}), \quad x \leq x_{\min} \quad (2.6)$$

while

$$x(t) = (x + x_{\min})(1 + O(Ut x^{-(\kappa-1)/\kappa})) - x_{\min} = x + O(Ut x^{1/\kappa}), \quad x \geq x_{\min} \quad (2.7)$$

Though we still get two different regimes, it makes sense to say that the long-time regime has been 'swallowed' by the short-time regime.

Summarizing, we get:

Lemma 2.4 *Let $t \geq 0$ and $x \in \mathbb{R}^d$.*

(1) (small cut-off regime) *Let $x_{\min} \in [0, (Ut)^{\kappa/(\kappa-1)}]$. Then $B_{\kappa, U, x_{\min}}(t, x) \subset B(x, C(\Phi_{\kappa, U}(t, |x|) - |x|))$ for some constant $C \geq 1$. Furthermore, there exists some constant $C' \geq 1$ such that, independently of x_{\min} :*

(i) *if $|x| \lesssim (Ut)^{\kappa/(\kappa-1)}$ (long-time regime),*

$$B_{\kappa, U, x_{\min}}(t, x) \subset B(0, C'(Ut)^{\kappa/(\kappa-1)}); \quad (2.8)$$

(ii) *if $|x| \gtrsim (Ut)^{\kappa/(\kappa-1)}$ (short-time regime),*

$$B_{\kappa, U, x_{\min}}(t, x) \subset B(x, C'(Ut)|x|^{1/\kappa}). \quad (2.9)$$

(2) (large cut-off regime) *There exists some constants $0 < c < 1 < C$ such that the following holds. Let $x_{\min} \geq (Ut)^{\kappa/(\kappa-1)}$. Then*

$$B(x, cUt \max(x_{\min}, |x|)^{1/\kappa}) \subset B_{\kappa, U, x_{\min}}(t, x) \subset B(x, CUt \max(x_{\min}, |x|)^{1/\kappa}). \quad (2.10)$$

Note the following particular case of (2.10),

$$B(x, c(Ut)^{\kappa/(\kappa-1)}) \subset B_{\kappa, U, (Ut)^{\kappa/(\kappa-1)}}(t, x) \subset B(x, C(Ut)^{\kappa/(\kappa-1)}), \quad |x| \leq (Ut)^{\kappa/(\kappa-1)}. \quad (2.11)$$

Remark 2.5 *In particular, an ode with velocity*

$$|v(s, y)| \lesssim U \left(1 + |y| + O(\sqrt{t})\right)^{1/\kappa} \quad (2.12)$$

is covered by Lemma 2.4 (1) for $t \geq U^{-1}$ since

$$\sup_{t \geq U^{-1}} \sqrt{t}/(Ut)^{\kappa/(\kappa-1)} = U^{-1/2} \leq 1. \quad (2.13)$$

Perturbation in $O(\sqrt{t})$ do appear as an effect due to diffusion (see §2.3). Thus the general philosophy is that convection prevails over diffusion in our setting.

2.2 The non-viscous case

We set the viscosity η to 0 in this paragraph. Namely, the zero-viscosity case is interesting in itself, easier to study, and contains already the main features of the viscous case (see §2.3 below). We are thus led to consider the approximation scheme

$$\phi^{(-1)} := 0; \quad (2.14)$$

$$(\partial_t + \phi^{(m-1)}(t, x) \cdot \nabla) \phi^{(m)}(t, x) = 0, \quad \phi^{(m)}|_{t=0} = u_0 \quad (m \geq 0) \quad (2.15)$$

to the non-viscous Burgers equation

$$(\partial_t + \phi \cdot \nabla)\phi = 0, \quad \phi|_{t=0} = u_0 \quad (2.16)$$

with initial condition u_0 satisfying (Hyp1). The zero-viscosity Feynman-Kac expression for the solution (compare with (1.4), (1.5)) is given in terms of deterministic characteristics $x^{(m)}(\cdot, x)$, $m \geq 0$, viz.

$$\phi^{(m)}(t, x) = u_0(x^{(m)}(t, x)), \quad (2.17)$$

where $x^{(m)}(t, x) := x^{(m)}(t; t, x)$ is the solution at time t of the ode

$$\begin{aligned} \frac{d}{ds} x^{(m)}(t; s, x) &= \phi^{(m-1)}(t - s, x^{(m)}(t; s, x)) \\ &= u_0(x^{(m-1)}(t - s, x^{(m)}(t; s, x))) \end{aligned} \quad (2.18)$$

with initial condition $x^{(m)}(t; 0, x) = x$. (Later on – see section 3 – we shall check inductively that $\phi^{(m)}(t, x)$ is continuous in time and Lipschitz in x , so that (2.18) has a unique solution, possibly only for small time.)

In particular,

$$x^{(0)}(t, x) = x; \quad (2.19)$$

$$\frac{d}{ds} x^{(1)}(t; s, x) = u_0(x^{(1)}(t; s, x)). \quad (2.20)$$

The ode for $x^{(1)}$ has by (Hyp1) a velocity bounded by $U(1 + |\cdot|)^{1/\kappa}$, so, by Definition 2.3,

$$x^{(1)}(t; s, x) \in B_{\kappa, U}(t, x), \quad s \leq t. \quad (2.21)$$

Then

$$\frac{d}{ds} x^{(2)}(t; s, x) = u_0(x^{(1)}(t - s, x^{(2)}(t; s, x))) \in u_0(B_{\kappa, U}(t, x^{(2)}(t; s, x))). \quad (2.22)$$

This suggests considering generalizations of the flow $t \mapsto \Phi_{\kappa, U}(t, x)$ of the following kind:

Definition 2.6 (generalized flow) *Let $t, x_{\min} > 0$ and $\kappa > 1$, $\tilde{U} \geq 1$. A generalized flow with initial velocity u_0 and parameters $(\kappa, \tilde{U}, x_{\min})$ (in short, a $(\kappa, \tilde{U}, x_{\min})$ -flow with velocity u_0 , or simply a $(\kappa, \tilde{U}, x_{\min})$ -flow if u_0 is clear from the context) is a system of ode's started from $x \in \mathbb{R}^d$,*

$$\frac{d}{ds} x(t; s, x) = u_0(X(t; s, x(t; s, x))), \quad x(t; 0, x) = x \quad (2.23)$$

with velocity field $v(t; s, \cdot) = u_0(X(t; s, \cdot))$ depending on the time-parameter t , such that $X(t; s, y) \in B_{\kappa, \tilde{U}, x_{\min}}(t, y)$, $y \in \mathbb{R}^d$.

The mapping $(s, y) \mapsto X(t; s, y)$ is simply called the mapping associated to the generalized flow (2.23).

Since our estimates concerning $(\kappa, \tilde{U}, x_{\min})$ -flows do not depend on x_{\min} provided $x_{\min} \leq (\tilde{U}t)^{\kappa/(\kappa-1)}$ (see Lemma 2.4), it is reasonable to assume that $x_{\min} \geq (\tilde{U}t)^{\kappa/(\kappa-1)}$ in the above Definition.

In the sequel, U is a fixed parameter associated to the growth at infinity of the initial velocity u_0 , while we let \tilde{U} vary in some range included in $[U, +\infty)$.

Under (Hyp1) such flows may be bounded very easily:

Lemma 2.7 *There exists some constant $C \geq 1$ such that the following holds. Let $\tilde{U} \geq U \geq 1$, $t \geq \tilde{U}^{-1}$, and $X(t; \cdot, \cdot)$ be the mapping associated to a $(\kappa, \tilde{U}, x_{\min})$ -flow. Assume the initial velocity u_0 satisfies (Hyp1). Then $x(t; s, x) \in B_{\kappa, CU, Ch_\kappa(t; \tilde{U}, x_{\min})}(t, x)$ for all $s \leq t$, where*

$$h_\kappa(t, \tilde{U}; x_{\min}) := \tilde{U}t \left(\max(x_{\min}, (\tilde{U}t)^{\kappa/(\kappa-1)}) \right)^{1/\kappa}. \quad (2.24)$$

Of course, this result holds for arbitrary small t provided one replaces $\tilde{U}t$ by $\langle \tilde{U}t \rangle$. Note the particular case,

$$h_\kappa(t, \tilde{U}; (\tilde{U}t)^{\kappa/(\kappa-1)}) = (\tilde{U}t)^{\kappa/(\kappa-1)}. \quad (2.25)$$

Proof. Clearly we may replace x_{\min} by $\max(x_{\min}, (\tilde{U}t)^{\kappa/(\kappa-1)})$. Hence we assume $x_{\min} \geq (\tilde{U}t)^{\kappa/(\kappa-1)}$ is a large cut-off, and use Lemma 2.4 (2) in the following form,

$$|X(t; s, y) - y| \leq C\tilde{U}t \max(x_{\min}, |y|)^{1/\kappa}. \quad (2.26)$$

We distinguish two cases:

(i) ($|y| \leq x_{\min}$) By (Hyp1)

$$\begin{aligned} |u_0(X(t; s, y))| &\leq U \left(1 + |y| + C\tilde{U}t x_{\min}^{1/\kappa} \right)^{1/\kappa} \\ &= U \left(1 + |y| + Ch_\kappa(t, U; x_{\min}) \right)^{1/\kappa}; \end{aligned} \quad (2.27)$$

(ii) ($|y| \geq x_{\min}$) By (Hyp1) again

$$\begin{aligned} |u_0(X(t; s, y))| &\leq U \left(1 + |y| + C\tilde{U}t |y|^{1/\kappa} \right)^{1/\kappa} \\ &\leq UC^{1/\kappa} (1 + |y| + \tilde{U}t |y|^{1/\kappa})^{1/\kappa} \leq U(2C)^{1/\kappa} (1 + |y|)^{1/\kappa} \leq CU(1 + |y|)^{1/\kappa} \end{aligned} \quad (2.28)$$

for C large enough;

which proves the Lemma. \square

In particular we have proved: $x^{(2)}(t; s, x) \in B_{\kappa, CU, C(Ut)^{\kappa/(\kappa-1)}}(t, x)$ for all $s \leq t$.

We may now iterate, and get for $m \geq 0$ and $t \geq U^{-1}$, using (2.25),

$$x^{(m)}(t; s, x) \in B_{\kappa, CU, x_{\min}^{(m)}}(t, x), \quad s \leq t \quad (2.29)$$

with $x_{\min}^{(0)} = x_{\min}^{(1)} = 0$, $x_{\min}^{(2)} = C(Ut)^{\kappa/(\kappa-1)}$, and

$$x_{\min}^{(m+1)} = Ch_\kappa(t, CU; x_{\min}^{(m)}) = C^2 U t (x_{\min}^{(m)})^{1/\kappa}, \quad m \geq 2. \quad (2.30)$$

This increasing recursive sequence converges for $m \rightarrow \infty$ for all $\kappa > 1$; we get by Lemma 5.1 a uniform bound for all $m \geq 0$,

$$x_{\min}^{(m)} \leq x_{\min}^{(\infty)} \quad (2.31)$$

where $x^{(\infty)} \lesssim (Ut)^{\kappa/(\kappa-1)}$ is the fixed point of the sequence.

All this strongly suggests that the approximation scheme should converge under the hypothesis (Hyp1). Leaving any rigor at this stage, and letting $m \rightarrow \infty$, one may conjecture that the solution of Burgers' equation satisfies for $t \geq U^{-1}$

$$u(t, x) \in u_0 \left(B_{\kappa, CU, C(Ut)^{\kappa/(\kappa-1)}}(t, x) \right). \quad (2.32)$$

Assuming (Hyp1), we get, using (2.27) and (2.28),

$$|u(t, x)| \lesssim U(|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{1/\kappa}. \quad (2.33)$$

Note however that, contrary to (2.31), this bound strongly relies on (Hyp1). When we consider later on more general initial conditions, (2.33) will be replaced by a much weaker bound, see (3.44) in Section 3.

2.3 The viscous case

We now come back to non-zero viscosity; we fix for simplicity $\eta = 1$. Instead of (2.18), we consider the approximation scheme (1.3) and its Feynman-Kac solution (1.4, 1.5). To avoid dealing with stochastic calculus tools we replace the stochastic differential equation (1.5) with an ode with random coefficients by letting $Y^{(m)}(t; s, x) := X^{(m)}(t; s, x) - B_s$, a conventional trick which is sometimes called the Doss-Sussmann trick: we thus get

$$\begin{aligned} \frac{d}{ds} Y^{(m)}(t; s, x) &= u^{(m-1)}(t-s, Y^{(m)}(t; s, x) + B_s) \\ &= \tilde{\mathbb{E}} \left[u_0(X^{(m-1)}(t-s, Y^{(m)}(t; s, x) + B_s)) \right] \\ &= \tilde{\mathbb{E}} \left[u_0(\tilde{B}_{t-s} + Y^{(m-1)}(t-s, Y^{(m)}(t; s, x) + B_s)) \right] \end{aligned} \quad (2.34)$$

where $X^{(m-1)}(t-s, y) = \tilde{B}_{t-s} + Y^{(m-1)}(t-s, y)$ is a random characteristic depending on an extra Wiener process $(\tilde{B}_t)_{t \geq 0}$, independent from B , and $\tilde{\mathbb{E}}[\cdot]$ is the partial expectation with respect to \tilde{B} . From standard results on Brownian motion, $\sup_{0 \leq s \leq t} |\tilde{B}_s|$ scales like \sqrt{t} and is actually bounded by $O(\sqrt{t})$ with high probability, namely, there exists a constant $c > 0$ such that $\mathbb{P}[\sup_{0 \leq s \leq t} |\tilde{B}_s| > A\sqrt{t}] \lesssim e^{-cA^2}$ for all $A > 0$. In the ensuing discussion we introduce the rescaled random variables,

$$M_t := 1 + \frac{\sup_{0 \leq s \leq t} |B_s|}{\sqrt{t}}, \quad \tilde{M}_t := 1 + \frac{\sup_{0 \leq s \leq t} |\tilde{B}_s|}{\sqrt{t}} \quad (2.35)$$

which are therefore $O(1)$ with high probability. In particular, for all $\alpha, A \geq 1$,

$$\mathbb{E}[(M_t)^\alpha] = O(1) \quad (2.36)$$

with a constant depending on α ,

$$\mathbb{P}[M_t > A] \lesssim e^{-cA^2} \quad (2.37)$$

for some universal constant c , and similarly for \tilde{M}_t .

Let us consider for the sake of illustration the cases $m = 0, 1$. First

$$Y^{(0)}(t; s, x) = x; \quad (2.38)$$

solving explicitly the trivial 0-th transport equation $(\partial_t - \Delta)u^{(0)}(t, x) = 0$, we get

$$\begin{aligned} \frac{d}{ds}Y^{(1)}(t; s, x) &= \tilde{\mathbb{E}}[u_0(Y^{(1)}(t; s, x) + B_s + \tilde{B}_{t-s})] \\ &= u^{(0)}(t-s; Y^{(1)}(t; s, x) + B_s) = e^{(t-s)\Delta}u_0(Y^{(1)}(t; s, x) + B_s). \end{aligned} \quad (2.39)$$

It is easy to check that

$$e^{t\Delta}(y \mapsto (1 + |y|)^{1/\kappa})(x) \lesssim 1 + t^{1/2\kappa} + |x|^{1/\kappa}. \quad (2.40)$$

Thus

$$\left| \frac{d}{ds}Y^{(1)}(t; s, x) \right| \lesssim U \left(1 + t^{1/2\kappa} + |B_s|^{1/\kappa} + |Y^{(1)}(t; s, x)|^{1/\kappa} \right). \quad (2.41)$$

Note that the same result may be retrieved without solving for $u^{(0)}$: namely,

$$\begin{aligned} \left| \tilde{\mathbb{E}}[u_0(Y^{(1)}(t; s, x) + B_s + \tilde{B}_{t-s})] \right| &\leq U \left\{ \tilde{\mathbb{E}} \left[1 + |\tilde{B}_{t-s}| + |B_s| + |Y^{(1)}(t; s, x)| \right] \right\}^{1/\kappa} \\ &\leq U \left(1 + M_t \sqrt{t} + |Y^{(1)}(t; s, x)| \right)^{1/\kappa} \end{aligned} \quad (2.42)$$

where we have used Jensen's inequality.

Hence (by definition) $Y^{(1)}(t; s, x) \in B_{\kappa, U, 1+M_t \sqrt{t}}$, implying in particular

$$Y^{(1)}(t; s, x) \in B_{\kappa, CU, \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, M_t \sqrt{t})}(t, x), \quad (2.43)$$

with the advantage that the cut-off is always *large* in this expression, in the sense of Lemma 2.7 (2). We may distinguish two regimes:

- (i) $M_t \sqrt{t} > \langle Ut \rangle^{\kappa/(\kappa-1)}$ (*diffusion prevails over convection*) then $Y^{(1)}(t; s, x) \in B_{\kappa, CU, M_t \sqrt{t}}(t, x)$, hence $|Y^{(1)}(t; s, x) - x| \lesssim Ut \max(|x|, M_t \sqrt{t})^{1/\kappa}$.

This case (i) is highly improbable if $U \gg 1$ (i.e. when convection effects are important) since

$$(M_t \sqrt{t} \gtrsim \langle Ut \rangle^{\kappa/(\kappa-1)}) \implies (M_t \gtrsim U^{1/2} \langle Ut \rangle^{\frac{1}{2} \frac{\kappa+1}{\kappa-1}} \geq U^{1/2}) \quad (2.44)$$

both if $t \leq U^{-1}$ and $t \geq U^{-1}$. For t large enough (depending on the random variable M_t) one is necessarily in case (ii);

- (ii) $M_t \sqrt{t} \lesssim \langle Ut \rangle^{\kappa/(\kappa-1)}$ (*convection prevails over diffusion*), then we simply get $Y^{(1)}(t; s, x) \in B_{\kappa, C'U, (C' \langle Ut \rangle)^{\kappa/(\kappa-1)}}(t, x)$.

As in the non-viscous case, we want to iterate. To go further, we need a rather straightforward adaptation to the viscous case of the notion of generalized $(\kappa, \tilde{U}, x_{\min})$ -flow introduced in the previous paragraph.

Definition 2.8 (viscous generalized flow) (compare with Definition 2.6) Let $\kappa > 0$ and $\kappa > 1$, $\tilde{U} \geq 1$. A viscous generalized flow with initial velocity u_0 and parameters $(\kappa, \tilde{U}, x_{\min})$ (in short, a viscous $(\kappa, \tilde{U}, x_{\min})$ -flow with velocity u_0 , or simply a viscous $(\kappa, \tilde{U}, x_{\min})$ -flow if u_0 is clear from the context) is a system of ode's with random coefficients started from $x \in \mathbb{R}^d$,

$$\frac{d}{ds}Y(t; s, x) = \tilde{\mathbb{E}}[u_0(\tilde{B}_{t-s} + \mathcal{Y}(t; s, Y(t; s, x) + B_s))], \quad Y(t; 0, x) = x \quad (2.45)$$

with random velocity field $v(t; s, \cdot) = \tilde{\mathbb{E}}[u_0(\tilde{B}_{t-s} + \mathcal{Y}(t; s, \cdot + B_s))]$ depending on the time-parameter t , such that $\mathcal{Y}(t; s, y) \in B_{\kappa, \tilde{U}, X_{\min}}(t, y)$, $y \in \mathbb{R}^d$, where $X_{\min} = X_{\min}(t)$ is a random variable depending on $(\tilde{B}_s)_{s \in [0, t]}$.

The mapping $(s, y) \mapsto \mathcal{Y}(t; s, y)$ is called the mapping associated to the viscous generalized flow (2.45).

In the above example, see (2.43), $X_{\min} = C \max((Ut)^{\kappa/(\kappa-1)}, M_t \sqrt{t})$.

Lemma 2.7 generalizes under (Hyp1) to the viscous case in the following way.

Lemma 2.9 *There exists some constant $C \geq 1$ such that the following holds. Let $t \geq \tilde{U}^{-1}$ and $\mathcal{Y}(t; \cdot, \cdot)$ be the mapping associated to a viscous generalized $(\kappa, \tilde{U}, \max(x_{\min}, \tilde{M}_t \sqrt{t}))$ -flow, with $x_{\min} \geq (\tilde{U}t)^{\kappa/(\kappa-1)}$ deterministic. Assume the initial velocity u_0 satisfies (Hyp1). Then*

$$\mathcal{Y}(t; s, x) \in B_{\kappa, CU, \max(C h_{\kappa}(t, \tilde{U}; x_{\min}), M_t \sqrt{t})}(t, x) \quad (2.46)$$

for all $s \leq t$, where $h_{\kappa}(t, \tilde{U}; x_{\min}) := \tilde{U}t x_{\min}^{1/\kappa}$, as in Lemma 2.7.

As in Lemma 2.7, we note that this result holds for arbitrary small t provided one replaces $\tilde{U}t$ by $\langle \tilde{U}t \rangle$.

Comparing with Lemma 2.7, one sees that the cut-off is larger due to diffusion in the highly improbable regime, defined by $M_t \sqrt{t} > x_{\min}$, where diffusion prevails over convection.

Proof. We distinguish two regimes:

(i) $(|y + B_s| \leq \max(x_{\min}, \tilde{M}_t \sqrt{t}))$. Then

$$|u_0(\tilde{B}_{t-s} + \mathcal{Y}(t; s, y + B_s))| \leq U \left(1 + \tilde{M}_t \sqrt{t} + |y + B_s| + C \tilde{U}t \left(\max(x_{\min}, \tilde{M}_t \sqrt{t}) \right)^{1/\kappa} \right)^{1/\kappa}, \quad (2.47)$$

whence (using $x_{\min}/\sqrt{t} \geq (Ut)^{\kappa/(\kappa-1)}/\sqrt{t} \geq U^{1/2} \geq 1$, see (2.13))

$$\begin{aligned} & \left| \tilde{\mathbb{E}} \left[\mathbf{1}_{|y+B_s| \leq \max(x_{\min}, \tilde{M}_t \sqrt{t})} u_0(\tilde{B}_{t-s} + \mathcal{Y}(t; s, y + B_s)) \right] \right| \\ & \lesssim U \left(1 + \sqrt{t} + |y + B_s| + \tilde{U}t (\max(x_{\min}, \sqrt{t}))^{1/\kappa} \right)^{1/\kappa} \\ & \lesssim U \left(|y| + M_t \sqrt{t} + \tilde{U}t x_{\min}^{1/\kappa} \right)^{1/\kappa} \end{aligned} \quad (2.48)$$

as expected;

(ii) $(|y + B_s| \geq \max(x_{\min}, \tilde{M}_t \sqrt{t}))$. Then

$$\begin{aligned} |u_0(\tilde{B}_{t-s} + \mathcal{Y}(t; s, y + B_s))| & \lesssim U \left(1 + \tilde{M}_t \sqrt{t} + |y + B_s| + \tilde{U}t |y + B_s|^{1/\kappa} \right)^{1/\kappa} \\ & \lesssim U |y + B_s|^{1/\kappa} \lesssim U (|y| + M_t \sqrt{t})^{1/\kappa}; \end{aligned} \quad (2.49)$$

which proves the Lemma.

□

Iterating as in the non-viscous case, we get for $m \geq 0$ and $t \geq U^{-1}$

$$Y^{(m)}(t, x) \in B_{\kappa, CU, \max(x_{\min}^{(m)}, M_t \sqrt{t})}(t, x) \quad (2.50)$$

with as in the non-viscous case, see (2.30),

$$x_{\min}^{(0)} = x_{\min}^{(1)} = 0, \quad x_{\min}^{(2)} = C(Ut)^{\kappa/(\kappa-1)}, \quad x_{\min}^{(m+1)} = Ch_{\kappa}(t, CU; x_{\min}^{(m)}) = C^2 Ut(x_{\min}^{(m)})^{1/\kappa} \quad (2.51)$$

bounded uniformly in m by $O((Ut)^{\kappa/(\kappa-1)})$.

Assuming as in the non-viscous case that the approximation scheme converges, it is natural to conjecture that the solution of Burgers' equation satisfies, still under (Hyp1)

$$\begin{aligned} |u(t, x)| &= \left| \lim_{m \rightarrow \infty} \mathbb{E}[u_0(X^{(m)}(t, x))] \right| \\ &\lesssim U \mathbb{E} \left[\left(|x| + M_t \sqrt{t} + (Ut)^{\kappa/(\kappa-1)} \right)^{1/\kappa} \right] \\ &\lesssim U(|x| + (Ut)^{\kappa/(\kappa-1)})^{1/\kappa} \end{aligned} \quad (2.52)$$

(see proof of Lemma 2.9) as in the non-viscous case.

3 More general initial data

From the previous section, in particular, Lemmas 2.7 and 2.9, it is reasonable to expect that the sequence $(u^{(m)})_{m \geq 0}$ is controlled as soon as flows driven by u_0 , or the 'generalized flows' thereof introduced in Definition 2.6, 2.8, are controlled well enough, in particular for t large, so as to ensure the possibility of an induction. This opens the way to flows subject to sudden but brief accelerations, corresponding to small areas where u_0 may be indeed very large; those must be brief enough so as not to change the behaviour of the flow for t large. What 'large' means is not so clear. Here we are interested in the whole regime $t \in [\frac{1}{U}, +\infty)$.

It would be natural to think of *defining* u_0 to be *admissible* if Lemmas 2.7 and 2.9, or some generalization thereof, hold. We did not find however any class of examples of admissible initial velocities u_0 which do not satisfy (Hyp1). Instead, we shall construct in the following way explicit examples of initial velocities for which we get uniform a priori bounds for the characteristics. First we consider some \tilde{u}_0 satisfying (Hyp1). Then we modify it in an essentially arbitrary way in a region with small relative volume, from which it can therefore escape in arbitrarily short time. The main challenge is to prove that there exist *safe zones*, with relative volume tending to 1 at spatial infinity, which are essentially *stable* under the flows – deterministically in the non-viscous case, with high probability in the viscous case. This *safe zone stability property* (see Theorem 3.1, Theorem 3.2) must be proved by induction. Then the complementary of the safe zones is made of small, widely separated islands, called *dangerous zones*, which by the safe zone stability property cannot communicate with each other; this simple fact settles non-inductively the analysis of trajectories started outside safe zones.

Let us mention that for a given velocity u_0 such that the associated flow has a relatively simple large scale topological structure (including large limit cycles, etc.) is not too complicated, the existence of large safe zones should not be too complicated to verify if true. Thus criteria (3.1, 3.2) below should merely be considered as some option.

Definition 3.1 Let $(R_n)_{n \geq 1}$ be an increasing sequence, $1 \leq R_1 < R_2 < R_3 < \dots$ such that, for all $i \geq 1$,

$$R_{2i} - R_{2i-1} \leq R_{2i-1}^{1/\kappa}, \quad (3.1)$$

$$R_{2i+1} \geq 4R_{2i}. \quad (3.2)$$

Annuli $B(0, R_{2i+1}) \setminus B(0, R_{2i})$ are called safe zones. Annuli $\mathcal{A}_i := B(0, R_{2i}) \setminus B(0, R_{2i-1})$ are called dangerous zones.

Remark 3.2 For convenience we repeatedly subdivide any large safe zone $B(0, R_{2i+1}) \setminus B(0, R_{2i})$ such that $R_{2i+1} \geq 16R_{2i}$ into $(B(0, 4R_{2i}) \setminus B(0, R_{2i})) \uplus \emptyset \uplus (B(0, R_{2i+1}) \setminus B(0, 4R_{2i}))$, with an empty dangerous zone sandwiched in-between, until all safe zones $B(0, R_{2i+1}) \setminus B(0, R_{2i})$ are such that $R_{2i+1} < 16R_{2i}$.

As explained in the introduction, our results hold if $R_{2i} - R_{2i-1} \leq CR_{2i-1}^{1/\kappa}$ and $R_{2i+1} \geq (1 + \varepsilon)R_{2i}$ for some $C, \varepsilon > 0$. We imposed (3.1, 3.2) because we did not want to make explicit the dependence of our bounds on C, ε .

We first consider the simpler non-viscous case.

3.1 Non-viscous case

To give a flavor of the proofs of Theorems 3.1 and 3.2 below, we start with the following elementary Lemma. It helps choosing a constant $C > 1$ such that

$$|y - x| \leq (C - 1)Ut \max(x_{\min}, |x|)^{1/\kappa} \quad (3.3)$$

provided $x_{\min} \geq (Ut)^{\kappa/(\kappa-1)}$ and $y \in B_{\kappa, U, x_{\min}}(t, x)$ (see Lemma 2.4 (2)).

In order to take into account various numerical constants coming from elementary estimates (Taylor expansions, etc.), **we assume once and for all that C is large enough.**

Lemma 3.3 Let $\tilde{u}_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an initial C^1 velocity satisfying (Hyp1) for some constants $U \geq 1$, $\kappa > 1$. Let $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be any Lipschitz function coinciding with \tilde{u}_0 outside the union of annuli $\cup_{i \geq 1} \mathcal{A}_i$, $\mathcal{A}_i := B(0, R_{2i}) \setminus B(0, R_{2i-1})$. Then the solution of the ode $\frac{dy}{ds} = u_0(y)$, $y(0) = x$, satisfies

$$|y(s) - x| \leq 16(C - 1)\langle Ut \rangle \max((16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, |x|)^{1/\kappa}, \quad 0 \leq s \leq t. \quad (3.4)$$

Proof.

Let us first make a general remark. If $u_0 \equiv \tilde{u}_0$ along the whole trajectory $(y(s))_{0 \leq s \leq t}$, then $y(s)$ is bounded as in (3.3), where we have set $x_{\min} = (Ut)^{\kappa/(\kappa-1)}$,

$$|y(s) - x| \leq (C - 1)Us \max((Ut)^{\kappa/(\kappa-1)}, |x|)^{1/\kappa}. \quad (3.5)$$

We must now distinguish two cases.

- (i) Let $|x| \geq (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$ (later on we shall actually need to assume that $|x| \geq 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$). Then $|x|^{1/\kappa} \leq |x|/16CUt$, so, provided $u_0 \equiv \tilde{u}_0$ along the whole trajectory,

$$|y(s)| \geq |x| - (C - 1)Ut|x|^{1/\kappa} \geq \frac{|x|}{2}, \quad |y(s)| \leq |x| + (C - 1)Ut|x|^{1/\kappa} \leq 2|x|. \quad (3.6)$$

Thus we check a posteriori that $u_0 \equiv \tilde{u}_0$ along the whole trajectory if

$$|x| \in I_i(t) := [R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}, R_{2i+1} - 4(C-1)UtR_{2i+1}^{1/\kappa}] \quad (3.7)$$

(with C large enough as stipulated above), with $R_{2i} \geq (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$; note that if $|x| \geq 16(16CUt)^{\kappa/(\kappa-1)}$ and (3.7) holds, then indeed $R_{2i} \geq \frac{1}{16}R_{2i+1} \geq (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$ by construction. Namely, if $|x| \in I_i(t)$ then

$$\begin{aligned} & (R_{2i+1} - 4(C-1)UtR_{2i+1}^{1/\kappa}) + (C-1)Us(R_{2i+1} - 4(C-1)UtR_{2i+1}^{1/\kappa})^{1/\kappa} \\ & \leq R_{2i+1} - 4(C-1)U(t-s)R_{2i+1}^{1/\kappa}; \end{aligned} \quad (3.8)$$

$$\begin{aligned} & (R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}) - (C-1)Us(R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa})^{1/\kappa} \\ & \geq R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa} - (C-1)Us(2R_{2i})^{1/\kappa} \\ & \geq R_{2i} + 4(C-1)U(t-s)R_{2i}^{1/\kappa} \end{aligned} \quad (3.9)$$

so

$$|y(s)| \in I_i(t-s) \subset I_i(0) = [R_{2i}, R_{2i+1}]. \quad (3.10)$$

We call $(I_i(t))_i$ *safe intervals*; (3.10) is the main argument in our *safe zone stability property*. Note that $I_i(t) \neq \emptyset$ since

$$(R_{2i+1} - 4(C-1)UtR_{2i+1}^{1/\kappa}) - (R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}) \geq \frac{1}{2}R_{2i+1} - \frac{3}{2}R_{2i} \geq \frac{1}{2}R_{2i} \quad (3.11)$$

by Hypothesis (3.2).

If now x does not belong to a safe zone, say, $|x| \in [R_{2i-1} - 4(C-1)UtR_{2i-1}^{1/\kappa}, R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}]$, then x is possibly free to move in essentially arbitrarily small time to $x' = y(t')$, $t' \in [0, t]$, such that $|x'|$ is the closest end of one of the two neighbouring safe zones, $I_j(t)$, with $j = i-1$ or i . Then for C large enough we get successively, using as unique ingredients Hypotheses (3.1, 3.2) and the lower bound $|x| \geq 16(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$,

$$R_{2i} \geq (8C\langle Ut \rangle)^{\kappa/(\kappa-1)};$$

$$R_{2i-1} \geq R_{2i} - O(R_{2i}^{1/\kappa}) \geq \frac{3}{4}R_{2i} \geq (6C\langle Ut \rangle)^{\kappa/(\kappa-1)};$$

$$|x| \geq R_{2i-1} - 4(C-1)UtR_{2i-1}^{1/\kappa} \geq \frac{R_{2i-1}}{3} \geq \frac{R_{2i}}{4};$$

$$\begin{aligned} |x' - x| & \leq (R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}) - (R_{2i-1} - 4(C-1)UtR_{2i-1}^{1/\kappa}) \\ & \leq (C-1)R_{2i-1}^{1/\kappa} \left\{ 1 + 2(C-1)UtR_{2i-1}^{(1/\kappa)-1} \right\} \\ & \leq 2(C-1)R_{2i-1}^{1/\kappa}; \end{aligned} \quad (3.12)$$

$$\begin{aligned} |x'| & \geq R_{2i} - |x' - x| \geq R_{2i} - 2CR_{2i-1}^{1/\kappa} \geq R_{2i} - 2CR_{2i}^{1/\kappa} \\ & \geq \frac{3}{4}R_{2i} \geq \frac{1}{2}(R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}) \geq \frac{|x|}{2}. \end{aligned} \quad (3.13)$$

Assume $|x| \geq 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$. If $j = i$ then $|x'| \geq |x|$ and $I_j(t) \subset [(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, \infty)$; otherwise $\min(I_j(0)) \geq \frac{1}{16} \max(I_j(t)) = \frac{|x'|}{16} \geq \frac{|x|}{32}$, so we get the same conclusion. Thus the rest of the trajectory (for $s \geq t'$) remains inside a safe zone and (3.6) holds, $|y(s) - x'| \leq (C-1)Ut|x'|^{1/\kappa}$. Hence for every $s \in [0, t]$, we get

$$\begin{aligned} |y(s) - x| &\leq 2(C-1)R_{2i-1}^{1/\kappa} + (C-1)Ut(R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa})^{1/\kappa} \\ &\leq 4(C-1)UtR_{2i}^{1/\kappa} \leq 16(C-1)Ut|x|^{1/\kappa}. \end{aligned} \quad (3.14)$$

Note that (3.14) improves on (3.4) in the initial time regime $Ut \leq 1$.

- (ii) Let $|x| \leq 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$. Then either the whole trajectory is contained in $B(0, 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)})$, or, letting $t' = \inf\{s \in [0, t] \mid |y(s)| = 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}\}$, we get by (i)

$$|y(s) - y(t')| \leq 16(C-1)Ut|y(t')|^{1/\kappa} \leq 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, \quad s \in [t', t] \quad (3.15)$$

hence in whole generality, $|y(s) - x| \leq 96(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$, $s \in [0, t]$.

□

Now comes the main result.

Theorem 3.1 (non-viscous case) *Let $\tilde{u}_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an initial C^1 velocity satisfying (Hyp1) for some constants $U \geq 1$, $\kappa > 1$. Let $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be any Lipschitz function coinciding with \tilde{u}_0 outside the union of annuli $\cup_{i \geq 1} \mathcal{A}_i$, $\mathcal{A}_i := B(0, R_{2i}) \setminus B(0, R_{2i-1})$. Then the sequence of characteristics $(x^{(m)}(t; \cdot, x))_{m \geq 0}$ satisfies the following uniform in m estimates:*

- (i) *Let $|x| \geq (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$, then $|x^{(m)}(t; s, x) - x| \lesssim (C-1)Ut|x|^{1/\kappa}$. If furthermore x is in a safe zone, $|x| \in I_i(t)$, such that $I_i(0) \subset [(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, \infty)$, then $|x^{(m)}(t; s, x)| \in I_i(t-s)$ for $0 \leq s \leq t$ (safe zone stability property).*
- (ii) *Let $|x| \leq (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$. Then $|x^{(m)}(t; s, x) - x| \lesssim (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$.*

Note that these estimates have just been proved in the case $m = 1$. We subdivide the **proof** into three points.

- (1) The core of the proof is the safe zone stability property. Let $i \geq 1$ such that $I_i(0) \subset [(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, \infty)$. Assume by induction that (see (3.6, 3.10))

$$(|x| \in I_i(t)) \implies \left(|x^{(m-1)}(t, x)| \in I_i(0), \frac{|x|}{2} \leq |x^{(m-1)}(t, x)| \leq 2|x| \right). \quad (3.16)$$

For such an x , we therefore know that in the ode for $x^{(m)}(t; \cdot, x)$,

$$\frac{d}{ds}y(s) = u_0(x^{(m-1)}(t-s, y(s))), \quad (3.17)$$

the norm of the argument of u_0 , $x^{(m-1)}(t-s, y(s))$, belongs to $I_i(0)$ provided $|y(s)| \in I_i(t-s)$. If this is the case, then

$$\left| \frac{d}{ds}y(s) \right| = |\tilde{u}_0(x^{(m-1)}(t-s, y(s)))| \leq U(1 + 2|y(s)|)^{1/\kappa} \leq 2^{1/\kappa}U(1 + |y(s)|)^{1/\kappa} \quad (3.18)$$

by our induction hypothesis (3.16), hence

$$\frac{|x|}{2} \leq |x| - 2^{1/\kappa}(C-1)Ut|x|^{1/\kappa} \leq |y(s)| \leq |x| + 2^{1/\kappa}(C-1)Ut|x|^{1/\kappa} \leq 2|x|. \quad (3.19)$$

This leads to a slight modification of (3.8,3.9),

$$\begin{aligned} & (R_{2i+1} - 4(C-1)UtR_{2i+1}^{1/\kappa}) + 2^{1/\kappa}(C-1)Us(R_{2i+1} - 4(C-1)UtR_{2i+1}^{1/\kappa})^{1/\kappa} \\ & \leq R_{2i+1} - 4(C-1)U(t-s)R_{2i+1}^{1/\kappa}; \end{aligned} \quad (3.20)$$

$$\begin{aligned} & (R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}) - 2^{1/\kappa}(C-1)Us(R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa})^{1/\kappa} \\ & \geq R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa} - 2^{1/\kappa}(C-1)Us(2R_{2i})^{1/\kappa} \\ & \geq R_{2i} + 4(C-1)U(t-s)R_{2i}^{1/\kappa} \end{aligned} \quad (3.21)$$

Hence we have checked a posteriori the safe zone stability property, $|y(s)| \in I_i(t-s)$.

- (2) Assume now $|x| \geq 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$ but x does not belong to a safe zone, say, $|x| \in [R_{2i-1} - 4(C-1)UtR_{2i}^{1/\kappa}, R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa}]$. From the proof of Lemma 3.3, we know that the trajectory, if ever, enters a safe interval $I_j(t)$, $j = i-1$ or i , at some point $x' = y(t')$ such that $|x'| \geq \frac{|x|}{2}$, and $I_j(0) \subset [(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, \infty)$. Hence we can avail ourselves of the safe zone stability property proved in (1), yielding $|y(t) - x'| \leq 2^{1/\kappa}(C-1)Ut|x|^{1/\kappa}$. Thus, for all $s \in [0, t]$,

$$\begin{aligned} |y(s) - x| & \leq 2(C-1)R_{2i-1}^{1/\kappa} + 2^{1/\kappa}(C-1)Ut(R_{2i} + 4(C-1)UtR_{2i}^{1/\kappa})^{1/\kappa} \\ & \leq 5(C-1)UtR_{2i}^{1/\kappa} \leq 20(C-1)Ut|x|^{1/\kappa}, \end{aligned} \quad (3.22)$$

as in (3.14), up to a different numerical constant.

- (3) Finally, for $|x| \leq 32(16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$, we conclude as in point (ii) of the proof of Lemma 3.3, again up to different numerical constants.

□

Under the hypotheses of Theorem 3.1, we obtain as in the previous section a conjectural uniform bound for $u^{(m)}$ and for u , which we write down for u ,

$$u(t, x) \in u_0(B_{\kappa, CU, C\langle Ut \rangle^{\kappa/(\kappa-1)}}(t, x)) \quad (3.23)$$

for some constant C , see (2.32), which is however not as explicit as (2.33).

3.2 Viscous case

Let us now consider the viscous case.

The new difficulty here is that, for $M_t \sqrt{t}$ or $\tilde{M}_t \sqrt{t}$ large, we clearly lose our safe zone stability property. Hence we need some general a priori bound on u_0 ; a polynomial bound at infinity is a very weak but sufficient requirement. Apart from that, the scheme follows closely that of §3.1.

Lemma 3.4 Let $\tilde{u}_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an initial velocity satisfying (Hyp1) for some constants $U \geq 1$, $\kappa > 1$. Let $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be any Lipschitz function coinciding with \tilde{u}_0 outside the union of annuli $\cup_{i \geq 1} \mathcal{A}_i$, $\mathcal{A}_i := B(0, R_{2i}) \setminus B(0, R_{2i-1})$ and satisfying the following a priori bound for some constants $\alpha, \beta \geq 0$,

$$|u_0(x)| \leq K_0(1 + |x|)^{\frac{\beta}{2} + \frac{1}{\kappa}}, \quad x \in \mathbb{R}^d \quad (3.24)$$

with

$$K_0 \leq U^{\frac{\beta}{2} + 1}. \quad (3.25)$$

Then the solution of the ode $\frac{d}{ds}Y(s) = \tilde{\mathbb{E}}[u_0(Y(s) + B_s + \tilde{B}_{t-s})]$, $Y(0) = x$ (see (2.39)), satisfies

$$|Y(s) - x| \lesssim (C - 1)\langle Ut \rangle \max\left((16C\langle Ut \rangle)^{\kappa/(\kappa-1)}, |x|\right)^{1/\kappa}, \quad (3.26)$$

if $M_t \sqrt{t} \leq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$,

$$|Y(s) - x| \lesssim \left(\frac{M_t \sqrt{t}}{\langle Ut \rangle}\right)^\kappa \quad (3.27)$$

if $M_t \sqrt{t} \geq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$.

The **proof** is a generalization of the non-viscous case, see proof of Lemma 3.3. We distinguish two regimes, (i) the *normal regime* where *convection prevails over diffusion* ($M_t \sqrt{t}$ small), and (ii) the regime where *diffusion prevails over convection* ($M_t \sqrt{t}$ large). The general idea is that the safe zone stability property holds in case (i), while the a priori bound (3.24) on u_0 yields new estimates in case (ii). Mind however (3.24) is also needed in case (i) since $\tilde{M}_t \sqrt{t}$ may be large. In particular (since a priori bounds alone would lead to a finite time explosion of the paths), $|y|$, U , t are controlled either *deterministically* by M_t – which is not averaged over here – or *stochastically* by \tilde{M}_t , when these get abnormally large.

As usual, we may in practice assume that $|x| \geq (16C\langle Ut \rangle)^{\kappa/(\kappa-1)}$.

(i) (normal regime) Assume $M_t \sqrt{t} \leq \langle Ut \rangle \langle x \rangle^{1/\kappa}$. We first need an a priori bound of

$$I^{(1)} := \left| \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{M}_t \sqrt{t} \geq \langle Ut \rangle \langle x \rangle^{1/\kappa}} u_0(Y(s) + B_s + \tilde{B}_{t-s}) \right] \right|. \quad (3.28)$$

The event $\tilde{\Omega} : \tilde{M}_t \sqrt{t} \geq \langle Ut \rangle \langle x \rangle^{1/\kappa}$ is a rare even of probability $O\left(\exp -c\left(\frac{\langle Ut \rangle}{\sqrt{t}} \langle x \rangle^{1/\kappa}\right)^2\right) = O(e^{-cUt} e^{-cU\langle x \rangle^{2/\kappa}})$ (the last equality holds both for $Ut \leq 1$ and $Ut \geq 1$!); thus $|x|$, but also U and t , are 'stochastically' controlled by \tilde{M}_t (see below). *Provided* $|Y(s)| \lesssim |x|$ we get

$$I^{(1)} \lesssim K_0 \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}} (|x| + \tilde{M}_t \sqrt{t})^{\frac{\beta}{2} + \frac{1}{\kappa}} \right]. \quad (3.29)$$

All factors in the above expression are highly suppressed by the exponentially small factors $O(e^{-cUt} e^{-cU\langle x \rangle^{2/\kappa}})$ since

$$K_0 \leq U^{\frac{\beta}{2} + 1} \leq (U\langle x \rangle^{2/\kappa})^{\frac{\beta}{2} + 1}, \quad |x| \lesssim (U\langle x \rangle^{2/\kappa})^{\kappa/2}, \quad \sqrt{t} \leq (Ut)^{1/2}. \quad (3.30)$$

Partitioning the event $\mathbf{1}_{\tilde{M}_t \sqrt{t} \geq \langle U_t \rangle \langle x \rangle^{1/\kappa}}$ into $\cup_{n \geq 0} \tilde{\Omega}_n$ where $\tilde{\Omega}_n := \{2^n \langle U_t \rangle \langle x \rangle^{1/\kappa} \leq \tilde{M}_t \sqrt{t} < 2^{n+1} \langle U_t \rangle \langle x \rangle^{1/\kappa}\}$, one can easily prove that, for c' small enough,

$$I \lesssim \sum_{n \geq 0} e^{-c' 2^n U \langle x \rangle^{2/\kappa}} \lesssim 1. \quad (3.31)$$

Hence, *provided* $|Y(s)| \approx |x|$,

$$\frac{d}{ds} Y(s) = O(1) + u_0(Y(s) + O(\langle U_t \rangle \langle x \rangle^{1/\kappa})) = O(1) + u_0(Y(s) + O(\langle U_t \rangle \langle Y(s) \rangle^{1/\kappa})). \quad (3.32)$$

Now, the innocuous replacement $Y(s) \mapsto Y(s) + O(\langle U_t \rangle \langle Y(s) \rangle^{1/\kappa})$ leaves the analysis of Lemma 3.3 unchanged, up to the following modifications: define

$$I_i(t) := [R_{2i} + 2(C-1)(\langle U_t \rangle + U_t)R_{2i}^{1/\kappa}, R_{2i+1} - 2(C-1)(\langle U_t \rangle + U_t)R_{2i+1}^{1/\kappa}] \quad (3.33)$$

(compare with (3.7)), so that the image of $I_i(0) = [R_{2i} + 2(C-1)R_{2i}^{1/\kappa}, R_{2i+1} - 2(C-1)R_{2i+1}^{1/\kappa}]$ by the mapping $y \mapsto y + O(\langle U_t \rangle \langle y \rangle^{1/\kappa})$ is $\subset [R_{2i}, R_{2i+1}]$. For C large enough and $|x| \in I_i(t)$, one gets $Y(s) \in [R_{2i} + 2(C-1)(\langle U_t \rangle + U(t-s))R_{2i}^{1/\kappa}, R_{2i+1} - 2(C-1)(\langle U_t \rangle + U(t-s))R_{2i+1}^{1/\kappa}] \subset I_i(t-s)$.

(ii) Assume on the contrary $M_t \sqrt{t} \geq \langle U_t \rangle \langle x \rangle^{1/\kappa}$; thus $\langle x \rangle$ is controlled in terms of $M_t \sqrt{t}$,

$$\langle x \rangle \leq \left(\frac{M_t \sqrt{t}}{\langle U_t \rangle} \right)^\kappa \leq (M_t \sqrt{t})^\kappa. \quad (3.34)$$

Thus the bound for $I^{(1)}$, see (i), is modified as follows *provided* $\langle Y(s) \rangle \leq \left(\frac{M_t \sqrt{t}}{\langle U_t \rangle} \right)^\kappa \leq (M_t \sqrt{t})^\kappa$,

$$\begin{aligned} I^{(1)} &\lesssim K_0 \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}} (|Y(s)| + \tilde{M}_t \sqrt{t} + M_t \sqrt{t})^{\frac{\alpha}{2} + \frac{1}{\kappa}} \right] \\ &\lesssim K_0 \left(|Y(s)|^{\frac{\alpha}{2} + \frac{1}{\kappa}} + \tilde{\mathbb{E}}[\mathbf{1}_{\tilde{\Omega}} (\tilde{M}_t \sqrt{t})^{\frac{\alpha}{2} + \frac{1}{\kappa}}] + (M_t \sqrt{t})^{\frac{\alpha}{2} + \frac{1}{\kappa}} \right) \\ &\lesssim U^{\frac{\beta}{2} + 1} \max(1, (M_t \sqrt{t})^\kappa)^{\frac{\alpha}{2} + \frac{1}{\kappa}} < \infty, \end{aligned} \quad (3.35)$$

to which one must add a smaller term,

$$I^{(1),c} := \left| \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}^c} u_0(Y(s) + B_s + \tilde{B}_{t-s}) \right] \right| \lesssim K_0 (|Y(s)|^{\frac{\alpha}{2} + \frac{1}{\kappa}} + (M_t \sqrt{t})^{\frac{\alpha}{2} + \frac{1}{\kappa}}). \quad (3.36)$$

Clearly (considering only powers of M_t for t fixed), these are very poor estimates of the velocity when $\frac{\alpha}{2} + \frac{1}{\kappa} > 1$, given the a priori condition $|Y(s)| = O(M_t^\kappa)$; actually we shall not need them.

Now, it may happen that $\langle Y(t') \rangle = \left(\frac{M_t \sqrt{t}}{\langle U_t \rangle} \right)^\kappa (\geq |x|)$ for some $t' \in [0, t]$. The estimates of (i) imply then in whole generality

$$|Y(s) - x| \lesssim \left(\frac{M_t \sqrt{t}}{\langle U_t \rangle} \right)^\kappa. \quad (3.37)$$

□

We may now state the main theorem of this section, a counterpart of Theorem 3.1 in the viscous case. Safe intervals are defined as in the previous lemma.

Theorem 3.2 (viscous case) Let $\tilde{u}_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an initial velocity satisfying (Hyp1) for some constants $U \geq 1, \kappa > 1$. Let $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be any Lipschitz function coinciding with \tilde{u}_0 outside the union of annuli $\cup_{i \geq 1} \mathcal{A}_i$, $\mathcal{A}_i := B(0, R_{2i}) \setminus B(0, R_{2i-1})$, and satisfying the a priori bounds (3.24). Then the sequence of characteristics $(Y^{(m)}(t; \cdot, x))_{m \geq 0}$ satisfies the following uniform in m estimates:

(i) (normal regime, $M_t \sqrt{t} \leq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$)

Then $|Y^{(m)}(t; s, x) - x| \lesssim (C - 1) \langle Ut \rangle \max((16C \langle Ut \rangle)^{\kappa/(\kappa-1)}, |x|)^{1/\kappa}$.

If furthermore x is in a safe zone, $|x| \in I_i(t)$, such that $I_i(t) \subset [(16C \langle Ut \rangle)^{\kappa/(\kappa-1)}, \infty)$, then, for all $x' \in \mathbb{R}^d$ such that $|x' - x| \leq \langle Ut \rangle \langle x \rangle^{1/\kappa}$ and all $y \in \mathbb{R}^d$ such that $|y - Y^{(m)}(t; s, x')| \leq \langle Ut \rangle \langle x \rangle^{1/\kappa}$, it holds $|y| \in I_i(t - s)$ (safe zone stability property).

(ii) Assume $M_t \sqrt{t} \geq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$. Then

$$|Y^{(m)}(t; s, x) - x| \lesssim \left(\frac{M_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa. \quad (3.38)$$

Proof. We proceed more or less as in the proof of Theorem 3.1. We cannot however separate the inductive proof of the safe zone stability property from the rest of the argument since we need the general bound (ii) to hold for $m - 1$ to control the contribution to the velocity of the event $\tilde{\Omega} : \tilde{M}_t \sqrt{t} \geq \max(Ut|x|^{1/\kappa}, (Ut)^{\kappa/(\kappa-1)})$. Thus we assume inductively that (i), (ii) hold for $m - 1$. As usual, we may restrict the study to $|x| \geq (16C \langle Ut \rangle)^{\kappa/(\kappa-1)}$.

(i) Assume first $M_t \sqrt{t} \leq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$ and let $x \in \mathbb{R}^d$ such that $|x| \in I_i(t)$, $I_i(t) \subset [(16C \langle Ut \rangle)^{\kappa/(\kappa-1)}, \infty)$. Recall $Y(s) := Y^{(m)}(t; s, x)$ solves the ode

$$\frac{d}{ds} Y(s) = \tilde{\mathbb{E}} \left[u_0(\tilde{B}_{t-s} + Y^{(m-1)}(t - s, Y(s) + B_s)) \right]. \quad (3.39)$$

If $\tilde{M}_t \sqrt{t} \leq \langle Ut \rangle \langle x \rangle^{1/\kappa}$, then $y := \tilde{B}_{t-s} + Y^{(m-1)}(t - s, Y(s) + B_s)$ satisfies precisely the assumptions of the safe zone stability property, hence $|y| \in I_i(0)$ provided $|Y(s)| \in I_i(t - s)$. Otherwise we first bound

$$I^{(m)} := \left| \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}} u_0(\tilde{B}_{t-s} + Y^{(m-1)}(t - s, Y(s) + B_s)) \right] \right|. \quad (3.40)$$

Provided $|Y(s)| \approx |x|$ we get by induction hypothesis

$$I^{(m)} \lesssim K_0 \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}} \left(|x| + \tilde{M}_t \sqrt{t} + \left(\frac{\tilde{M}_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa \right)^{\frac{\kappa}{2} + \frac{1}{\kappa}} \right] \lesssim 1 \quad (3.41)$$

as in Lemma 3.4. The rest of the argument is as in the non-viscous case (see proof of Theorem 3.1).

(ii) Assume now $M_t \sqrt{t} \geq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa})$. By induction hypothesis we get

$$\begin{aligned} I^{(m)} &\lesssim K_0 \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}} \left(|Y(s)| + M_t \sqrt{t} + \tilde{M}_t \sqrt{t} + \left(\frac{\tilde{M}_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa \right)^{\frac{\kappa}{2} + \frac{1}{\kappa}} \right] \\ &\lesssim U^{\frac{\kappa}{2} + 1} \max(1, (M_t \sqrt{t})^\kappa)^{\frac{\kappa}{2} + \frac{1}{\kappa}} < \infty \end{aligned} \quad (3.42)$$

to which we must add a smaller contribution,

$$\begin{aligned} I^{(m),c} &:= \left| \tilde{\mathbb{E}} \left[\mathbf{1}_{\tilde{\Omega}^c} u_0(\tilde{B}_{t-s} + Y^{(m-1)}(t-s, Y(s) + B_s)) \right] \right| \\ &\lesssim K_0 \left(|Y(s)| + M_t \sqrt{t} + \left(\frac{\tilde{M}_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa \right)^{\frac{\alpha}{2} + \frac{1}{\kappa}} \end{aligned} \quad (3.43)$$

as in (3.35,3.36). Using (i) one concludes as in (3.37): $|Y(s) - x| \lesssim \left(\frac{M_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa$.

□

Using the above Theorem we may conjecture that the following uniform bounds hold for $u^{(m)}$, $m \geq 0$ and for u ,

$$\begin{aligned} |u(t, x)| &= \left| \lim_{m \rightarrow \infty} \mathbb{E} \left[u_0(X^{(m)}(t, x)) \right] \right| \\ &\lesssim K_0 \mathbb{E} \left[\left(|x| + M_t \sqrt{t} + \langle Ut \rangle^{\kappa/(\kappa-1)} + \mathbf{1}_{M_t \sqrt{t} \geq \langle Ut \rangle \langle x \rangle^{1/\kappa}} \left(\frac{M_t \sqrt{t}}{\langle Ut \rangle} \right)^\kappa \right)^{\frac{\alpha}{2} + \frac{1}{\kappa}} \right] \\ &\lesssim K_0 (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\frac{\alpha}{2} + \frac{1}{\kappa}}. \end{aligned} \quad (3.44)$$

(see proof of Lemma 3.4 (i)).

4 Proof of the convergence of the scheme

The general assumptions on u_0 in this main section are:

(i) u_0 is a C^2 function;

(ii) (a priori bounds on $u_0, \nabla u_0, \nabla^2 u_0$) there exist constants $\alpha, \beta \geq 0$ such that, for all $x \in \mathbb{R}^d$,

$$|u_0(x)| \leq K_0(1 + |x|)^{\frac{\alpha}{2} + \frac{1}{\kappa}}, \quad |\nabla u_0(x)| \leq K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}, \quad |\nabla^2 u_0(x)| \leq K_2(1 + |x|)^{\frac{3}{2}(\frac{\alpha}{2} + \frac{1}{\kappa})} \quad (4.1)$$

with $K_0 \leq U^{\frac{\beta}{2}+1}, K_0 \leq K_1^{1/2}, U \leq K_1 \leq K_2^{2/3}$;

(iii) u_0 coincides outside the union of annuli $\cup_{i \geq 1} \mathcal{A}_i$ with an initial velocity \tilde{u}_0 satisfying (Hyp1),

annuli $(\mathcal{A}_i)_{i \geq 1}$ being as in Definition 3.1.

Note that this set of assumptions is precisely that of Theorem 3.2, plus some extra a priori bounds on $\nabla u_0, \nabla^2 u_0$. We let $M_t := 1 + \frac{\sup_{0 \leq s \leq t} |B_s|}{\sqrt{t}}$ as in the previous sections. Generalizing (iii), we may assume that the sequence of random characteristics $(Y^{(m)}(t; \cdot, x))_{m \geq 0}$ satisfies some weaker form of the conclusions of Theorem 3.2,

(iii)' random characteristics $(Y^{(m)}(\cdot; \cdot, \cdot))_{m \geq 0}$ obey the following estimates,

$$|Y^{(m)}(t; s, x) - x| \leq (C_\kappa - 1)\langle Ut \rangle \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, |x|)^{1/\kappa} \quad (4.2)$$

$$\text{if } M_t \sqrt{t} \leq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa});$$

$$|Y^{(m)}(t; s, x) - x| \lesssim (M_t \sqrt{t})^{\kappa'} \quad (4.3)$$

$$\text{if } M_t \sqrt{t} \geq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa}),$$

for some large enough constant $C_\kappa > 1$, and some exponent $\kappa' \geq 1$ possibly differing from κ ,

hypothesis (iii) or more generally (iii)' implying in turn a uniform in m bound on $u^{(m)}$,

$$|u^{(m)}(t, x)| \lesssim K_0(|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\frac{\alpha}{2} + \frac{1}{\kappa}} \quad (4.4)$$

(see (3.44)), which completes the proof of Theorem 1 in the Introduction.

We now proceed to prove by induction the bounds on $\nabla u^{(m)}, \nabla^2 u^{(m)}, v^{(m)}, \nabla v^{(m)}$ collected in Theorem 2 (see section 1.2). All subsequent computations rely exclusively on Feynman-Kac's formula, Schauder estimates, hypotheses (i),(ii), the bounds on the characteristics, (4.2,4.3), and their immediate corollary (4.4).

4.1 Scheme of proof

We first want to bound the gradient functions $\nabla u^{(m)}$, $m \geq 0$. By using the Feynman-Kac representation and the bounds on the characteristics (4.2, 4.3), it is easy in the non-viscous case to derive local a priori bounds for the gradient in some initial regime $t \leq T_{\min}(x)$; however, since $T_{\min}(x) \rightarrow 0$ when $|x| \rightarrow \infty$, one cannot draw from this fact alone any conclusion about global-in-space, local-in-time regularity of the solution. This works also fine in the viscous case *provided* $\alpha = 0$, i.e. u_0 is *sublinear* (or, in other words, *if (Hyp1) is verified*), and ∇u_0 *subquadratic*, because large deviation estimates (i.e. Gaussian bounds) for Brownian motion suffice to control the gradient for $t \leq T_{\min}(x)$. In the latter case, parabolic Schauder estimates (requiring a non-zero viscosity) make it possible to extend these bounds to arbitrarily large time. To deal with the general (viscous) case, we replace eq. (1.3) for $u^{(m)}$ by a family $u^{(m,n)}$ of *penalized* transport equations, meant as a smoothened substitute of the original equation solved on dyadic balls $B(0, 2^n), n \geq 0$ with Dirichlet boundary conditions. Gradient bounds for the solutions $u^{(m,n)}$ are easily obtained in some n -dependent initial regime $t \leq T_n(x)$, and again extended to later times thanks to Schauder estimates. Then we prove that the series $\sum_n |u^{(m,n)} - u^{(m,n-1)}|$ converges. The same techniques can be repeated to bound second derivatives $\nabla^2 u^{(m)}$ (see §4.2).

In turn we use the uniform estimates for $\nabla u^{(m)}$ found in §4.2, together with those for $u^{(m)}$ (see (4.4)) to bound $v^{(m)} := u^{(m)} - u^{(m-1)}$ by simple time integration. For fixed x , we obtain $v^{(m)}(t, x) = O\left(\left(K_1 \frac{t}{m}\right)^m\right)$ for $t = O(m/K_1)$ (called: *short-time regime*), $O(1)$ otherwise. Thus for fixed t, x , the series $\sum_m |v^{(m)}|$ converges locally uniformly (see §4.3).

Finally, repeating the techniques of §4.2, we bound $\nabla v^{(m)}$ and deduce that the series $\sum_m |\nabla v^{(m)}|$ converges locally uniformly (see §4.4). Thus the limit of the series is a solution of Burgers' equation.

Note that, by a standard argument using Schauder's estimates, the solution may be proved to be smooth for $t > 0$. If higher order derivatives of u_0 are polynomially bounded, then the regularity may be proved along the same lines to extend down to $t = 0$. In particular, the solution is classical if u_0 is C^2 .

4.2 Gradient bounds

We prove in this section the bounds (1.16), (1.17) on $\nabla u^{(m)}$ and $\nabla^2 u^{(m)}$.

4.2.1 Gradient bounds in the initial regime

By taking the gradient of (1.3), we get

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla + \nabla u^{(m-1)}) \nabla u^{(m)} = 0. \quad (4.5)$$

Note that $(\nabla u^{(m-1)})$ is a matrix with entries $(\nabla u^{(m-1)})_{ij} := \partial_i u_j^{(m-1)}$. Feynman-Kac formula implies the following representation of the solution,

$$\nabla u^{(m)}(t, x) = \mathbb{E} \left[T \left(e^{-\int_0^t \nabla u^{(m-1)}(t-s, X^{(m)}(t,s,x)) ds} \right) \nabla u_0(X^{(m)}(t, x)) \right] \quad (4.6)$$

where $T(\cdot)$ is the time-ordering operator, namely,

$$T \left(e^{\int_0^t B(s) ds} \right) := \sum_{n \geq 0} \int_{t > s_1 > \dots > s_n > 0} B(s_1) \dots B(s_n) ds_1 \dots ds_n \quad (4.7)$$

is the solution at time t of the matrix-valued ode $\frac{d}{dt} M(t) = B(t)M(t)$ started from the identity. We will be happy with the simple bound in terms of matrix norm $\|\cdot\|$, $\|M(t)\| \leq \exp \left(\int_0^t \|B(s)\| ds \right)$.

Let us illustrate this for $m = 0, 1$. First

$$\nabla u^{(0)}(t, x) = \mathbb{E}[\nabla u_0(x + B_t)] = e^{t\Delta} \nabla u_0(x) \quad (4.8)$$

hence (see (2.40), (2.44))

$$|\nabla u^{(0)}(t, x)| \lesssim K_1(1 + \sqrt{t} + |x|)^{\alpha + \frac{2}{\kappa}} \lesssim K_1(|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}}. \quad (4.9)$$

As in §2.3, this bound may also be found directly without using the explicit solution for $u^{(0)}$; namely,

$$|\nabla u^{(0)}(t, x)| \leq K_1 \mathbb{E}[(1 + |x| + |B_t|)^{\alpha + \frac{2}{\kappa}}] \lesssim K_1(1 + \sqrt{t} + |x|)^{\alpha + \frac{2}{\kappa}}. \quad (4.10)$$

Next, we consider the case $m = 1$. At this stage one readily understands that the representation (4.6) alone does not allow an inductive bound, uniform in m , of $\nabla u^{(m)}(t, x)$ for $t \leq T(x)$, where $T(x) > 0$ is any *deterministic* (possibly x -dependent) time. Namely, assuming $\kappa' \geq \kappa$ to make a case, the function in the time-ordered exponential scales for $m = 1$ roughly like

$$tK_1 \left(|x| + \langle Ut \rangle^{\kappa/(\kappa-1)} + (M_t \sqrt{t})^\kappa \right)^{\alpha + \frac{2}{\kappa}} \gtrsim F(t, M_t) := tK_1 (M_t \sqrt{t})^{2+\kappa\alpha} \quad (4.11)$$

for t small, i.e. $Ut \leq 1$, and M_t large, i.e. $M_t \sqrt{t} \geq \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, \langle Ut \rangle \langle x \rangle^{1/\kappa}) \approx (1 + |x|)^{1/\kappa}$. Hence $F(t, M_t)$ grows for fixed t roughly like M_t^γ , with $\gamma = 2 + \kappa\alpha > 2$ as soon as $\alpha > 0$, which gives seemingly an infinite average for the exponential factor (compare with Gaussian queue (2.35)). On the other hand (see more details below), we note that in the 'normal' regime where (assuming $Ut \leq 1$) $Y^{(1)}(t; s, x) \in B_{\kappa, CU}(t, x)$, $|Y^{(1)}(t; s, x)| \lesssim \langle x \rangle$, the function in the exponential scales roughly like $tK_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}$. By reference to this case we let, with $C > 1$ large enough

Definition 4.1 $T_{\min}(x) := \left(C^3 K_1 (1 + |x|)^{\alpha + \frac{2}{\kappa}} \right)^{-1}$

and terminate this somewhat sloppy discussion by some detailed computations.

Let $t \leq T_{\min}(x)$ (implying in particular $Ut \leq 1$ by Hypothesis (ii)), and $\Omega := \mathbf{1}_{M_{T_{\min}(x)} \geq \sqrt{K_1}(1+|x|)^{\frac{\alpha}{2} + \frac{2}{\kappa}}}$. On Ω^c one has $M_t \sqrt{t} \leq M_{T_{\min}(x)} \sqrt{T_{\min}(x)} \leq (1+|x|)^{1/\kappa} \approx \max(\langle UT_{\min}(x) \rangle^{\kappa/(\kappa-1)}, \langle UT_{\min}(x) \rangle \langle x \rangle^{1/\kappa})$, hence one is in the 'normal' regime where convection dominates over diffusion. Then $|Y^{(1)}(t; s, x) - x| \lesssim \langle Ut \rangle \max(\langle Ut \rangle^{\kappa/(\kappa-1)}, |x|)^{1/\kappa} \lesssim (1+|x|)^{1/\kappa}$, hence $|Y^{(1)}(t; s, x)|, |X^{(1)}(t; s, x)| \lesssim 1 + |x|$ as pointed out earlier, and

$$\int_0^t |\nabla u^{(0)}(t-s, X^{(1)}(t; s, x))| ds \lesssim t K_1 (1+|x|)^{\alpha + \frac{2}{\kappa}} \lesssim T_{\min}(x) K_1 (1+|x|)^{\alpha + \frac{2}{\kappa}} \leq 1 \quad (4.12)$$

for C large enough, as required. Similarly, $|\nabla u_0(X^{(1)}(t, x))| \lesssim K_1 (1+|x|)^{\alpha + \frac{2}{\kappa}}$. On the whole we have proved:

$$\begin{aligned} I^c &:= \left| \mathbb{E} \left[\mathbf{1}_{\Omega^c} T \left(e^{-\int_0^t \nabla u^{(0)}(t-s, X^{(1)}(t; s, x)) ds} \right) \nabla u_0(X^{(1)}(t, x)) \right] \right| \\ &\lesssim K_1 (1+|x|)^{\alpha + \frac{2}{\kappa}}, \end{aligned} \quad (4.13)$$

a bound comparable to the a priori bound (4.1) for u_0 .

However for the time being, we fall short of proving a bound for $|\nabla u^{(1)}(t, x)|$ for $t \leq T_{\min}(x)$ since we have disregarded the event Ω . The reason is that we have not used the *regularizing effect of diffusion*.

We henceforth develop a more comprehensive strategy of proof, incorporating parabolic Schauder estimates.

By induction we assume that for some large enough constant $C > 1$,

(Induction hypothesis)

$$|\nabla u^{(m-1)}(t, x)| \leq C^2 K_1 (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}}. \quad (4.14)$$

The constant C in (4.14) is the same as in the definition of $T_{\min}(x)$ (see Definition 4.1), and also the same as that appearing in the bounds for $\nabla^2 u^{(m)}$ (see (4.43)), $v^{(m)}$ (see (4.48)) and $\nabla v^{(m)}$ (see (4.55)). It should be large enough to satisfy various requirements turning up in the course of the proofs. The important point to be checked carefully is that it may be chosen *uniform in m* .

We fix some smooth function $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\chi|_{[0,1]} = 0$ and $\chi|_{[2,+\infty)} = 1$, and let $\chi^{(n)}(|x|) := \chi(2^{-n}|x|)$, $n \geq 0$.

Definition 4.2 (i) For $n \in \mathbb{N}$, let $u^{(m,n)} : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the solution of the transport equation $(\partial_t - \Delta + u^{(m-1)}(t, x) \cdot \nabla) u^{(m,n)}(t, x) = -2C^2 K_1 (2(1+|x|^2))^{\frac{\alpha}{2} + \frac{1}{\kappa}} \chi^{(n)}(|x|) u^{(m,n)}(t, x)$ with initial condition $u^{(m,n)}(t=0) = u_0$.

(ii) Let $\underline{u}^{(m,n)} := u^{(m,n)} - u^{(m,n-1)}$ ($n \geq 1$).

Let us write for short $F_n(x) := 2C^2 K_1 (2(1+|x|^2))^{\frac{\alpha}{2} + \frac{1}{\kappa}} \chi^{(n)}(|x|)$. The main properties of F_n are the following: $F_n(x) \geq 0$, F_n is smooth and:

- (i) $F_n(x) = 0$ if $|x| \leq 2^n$;
- (ii) $F_n(x) \geq 2C^2 K_1(1 + |x|)^{\alpha + \frac{2}{k}} \geq 2|\nabla u^{(m-1)}(t, x)|$ if $U t \leq 1$ and $|x| \geq 2^{n+1}$;
- (iii) $|\nabla F_n(x)| \lesssim C^2 K_1(1 + |x|)^{\alpha + \frac{2}{k} - 1} \mathbf{1}_{|x| \geq 2^n}$, $|\nabla^2 F_n(x)| \lesssim C^2 K_1(1 + |x|)^{\alpha + \frac{2}{k} - 2} \mathbf{1}_{|x| \geq 2^n}$.

As it happens (see below), the dampening of the solution for $|x|$ large is strong enough to ensure a rapid fall-off outside the ball $B(0, 2^n)$; compared to more conventional Dirichlet boundary conditions, this has the advantage of avoiding uncontrollable boundary effects.

The Feynman-Kac representation for $u^{(m,n)}$ is

$$u^{(m,n)}(t, x) = \mathbb{E} \left[u_0(X^{(m)}(t, x)) e^{-\int_0^t ds F_n(X^{(m)}(t, s, x))} \right]. \quad (4.15)$$

By subtracting, one gets

$$\underline{u}^{(m,n)}(t, x) = \mathbb{E} \left[\mathbf{1}_{X^{(m)}(t, \cdot, x) \notin B(0, 2^{n-1})} u_0(X^{(m)}(t, x)) \left(e^{-\int_0^t ds F_n(X^{(m)}(t, s, x))} - e^{-\int_0^t ds F_{n-1}(X^{(m)}(t, s, x))} \right) \right] \quad (4.16)$$

where $X^{(m)}(t; \cdot, x) := \{X^{(m)}(t; s, x), 0 \leq s \leq t\}$ is the image of the characteristic.

Differentiating, we get

$$(\partial_t - \Delta + u^{(m-1)} \cdot \nabla) \nabla u^{(m,n)}(t, x) = -(\nabla u^{(m-1)}(t, x) + F_n(x)) \nabla u^{(m,n)}(t, x) - \nabla F_n(x) u^{(m,n)}(t, x) \quad (4.17)$$

yielding the Feynman-Kac representation

$$\nabla \underline{u}^{(m,n)}(t, x) \equiv w_1^{(m,n)}(t, x) - \int_0^t ds \left(w_2^{(m,n)}(t, s, x) + w_3^{(m,n)}(t, s, x) + w_4^{(m,n)}(t, s, x) \right), \quad (4.18)$$

with (letting $X^{(m)}(t; \leq s, x) := \{X^{(m)}(t; s', x), 0 \leq s' \leq s\}$) :

$$\begin{aligned} w_1^{(m,n)}(t, x) &:= \mathbb{E} \left[\mathbf{1}_{X^{(m)}(t; \cdot, x) \notin B(0, 2^{n-1})} \left(e^{-\int_0^t ds F_n(X^{(m)}(t, s, x))} - e^{-\int_0^t ds F_{n-1}(X^{(m)}(t, s, x))} \right) \right. \\ &\quad \left. T \left(e^{-\int_0^t ds \nabla u^{(m-1)}(t-s, X^{(m)}(t, s, x))} \right) \nabla u_0(X^{(m)}(t, x)) \right] \end{aligned} \quad (4.19)$$

$$\begin{aligned} w_2^{(m,n)}(t, s, x) &:= \mathbb{E} \left[\mathbf{1}_{X^{(m)}(t; \leq s, x) \notin B(0, 2^{n-1})} \left(e^{-\int_0^s ds' F_n(X^{(m)}(t, s', x))} - e^{-\int_0^s ds' F_{n-1}(X^{(m)}(t, s', x))} \right) \right. \\ &\quad \left. T \left(e^{-\int_0^s ds' \nabla u^{(m-1)}(t-s', X^{(m)}(t, s', x))} \right) \nabla F_{n-1}(X^{(m)}(t, s, x)) u^{(m,n-1)}(t-s, X^{(m)}(t, s, x)) \right]; \end{aligned} \quad (4.20)$$

$$\begin{aligned} w_3^{(m,n)}(t, s, x) &:= \mathbb{E} \left[\mathbf{1}_{X^{(m)}(t; \leq s, x) \notin B(0, 2^{n-1})} e^{-\int_0^s ds' F_n(X^{(m)}(t, s', x))} \right. \\ &\quad \left. T \left(e^{-\int_0^s ds' \nabla u^{(m-1)}(t-s', X^{(m)}(t, s', x))} \right) \nabla (F_n - F_{n-1})(X^{(m)}(t, s, x)) u^{(m,n-1)}(t-s, X^{(m)}(t, s, x)) \right]; \end{aligned} \quad (4.21)$$

$$\begin{aligned} w_4^{(m,n)}(t, s, x) &:= \mathbb{E} \left[e^{-\int_0^s ds' F_n(X^{(m)}(t, s', x))} T \left(e^{-\int_0^s ds' \nabla u^{(m-1)}(t-s', X^{(m)}(t, s', x))} \right) \right. \\ &\quad \left. \nabla F_n(X^{(m)}(t, s, x)) \underline{u}^{(m,n)}(t-s, X^{(m)}(t, s, x)) \right], \end{aligned} \quad (4.22)$$

as deduced from the Feynman-Kac representation for $\nabla u^{(m,n)}(t, x)$,

$$\nabla u^{(m,n)}(t, x) \equiv u_1^{(m,n)}(t, x) - \int_0^t ds u_2^{(m,n)}(t; s, x), \quad (4.23)$$

where

$$u_1^{(m,n)}(t, x) := \mathbb{E} \left[e^{-\int_0^t ds F_n(X^{(m)}(t; s, x))} T \left(e^{-\int_0^t ds \nabla u^{(m-1)}(t-s, X^{(m)}(t; s, x))} \right) \nabla u_0(X^{(m)}(t, x)) \right] \quad (4.24)$$

$$u_2^{(m,n)}(t; s, x) := \mathbb{E} \left[e^{-\int_0^s ds' F_n(X^{(m)}(t; s', x))} T \left(e^{-\int_0^s ds' \nabla u^{(m-1)}(t-s', X^{(m)}(t; s', x))} \right) \nabla F_n(X^{(m)}(t; s, x)) u^{(m,n)}(t-s, X^{(m)}(t; s, x)) \right] \quad (4.25)$$

We shall now bound: $u^{(m,n)}(t, y)$, $\underline{u}^{(m,n)}(t, y)$ ($y \in \mathbb{R}^d$) for $t \leq T_{\min}(0) = (C^3 K_1)^{-1}$; and each of the terms contributing to $\nabla \underline{u}^{(m,n)}(t, x)$ for $\langle x \rangle \leq (2C_\kappa)^{-2} 2^{n-1}$ and $t < T_n$, where

$$T_n := \left(C^3 K_1 (2^n)^{\alpha + \frac{2}{\kappa}} \right)^{-1}, \quad n \geq 0. \quad (4.26)$$

Note that $T_n \approx T_{\min}(2^n)$.

The *main point* to be understood is that the events $(X^{(m)}(t; \leq s, x) \notin B(0, 2^{n-1}))$, figuring inside the expectations defining \underline{u} , w_1 , w_2 and w_3 , are extremely unlikely for n large. Namely, choose C_κ large enough; by hypothesis,

$$|X^{(m)}(t; s, x) - x| \leq |Y^{(m)}(t; s, x) - x| + M_t \sqrt{t} \leq (C_\kappa - 1) \langle x \rangle^{1/\kappa} + O((M_t \sqrt{t})^{\kappa'}) \quad (4.27)$$

for $t \leq U^{-1}$ (recall $\kappa' \geq 1$). From this we conclude: if $\langle x \rangle \leq (2C_\kappa)^{-1} 2^{n-1}$ (hence in particular, $2^n \geq 4C_\kappa \gg 1$), and $t \leq T_{\min}(0)$,

$$M_t \gtrsim \frac{(2^n)^{\kappa'}}{\sqrt{T_{\min}(0)}} \geq C^{3/2} \sqrt{K_1} (2^n)^{1/\kappa'}, \quad (4.28)$$

an event of probability $O(e^{-cC^3})O(e^{-cK_1})O(e^{-c(2^n)^{2/\kappa'}})$.

- (i) (bound for $u^{(m,n)}(t, x)$, $t \leq T_{\min}(0)$) We replace $u^{(m,n)}(t, x) = \mathbb{E}[\cdot]$ with $\mathbb{E}[\mathbf{1}_{X^{(m)}(t, x) \in B(0, 2^{n-1})} \cdot] + \sum_{p \geq n-1} \mathbb{E}[\mathbf{1}_{X^{(m)}(t, x) \in B(0, 2^{p+1}) \setminus B(0, 2^p)} \cdot]$. Since $|u_0(X^{(m)}(t, x))| \leq K_0(1 + |X^{(m)}(t, x)|)^{\frac{\alpha}{2} + \frac{1}{\kappa}}$, the first and main term is a $O(K_0(2^n)^{\frac{\alpha}{2} + \frac{1}{\kappa}})$. Subsequent terms are $\lesssim K_0(2^p)^{\frac{\alpha}{2} + \frac{1}{\kappa}} \cdot O(e^{-cK_1})O(e^{-c(2^p)^{2/\kappa'}}) \lesssim e^{-c'(2^p)^{2/\kappa'}}$, summing up to $O(1)$.

Let us also bound $u^{(m,n-1)}(t, y)$, with $y \in \mathbb{R}^d$ (see (4.21)). If $|y| \ll 2^n$, the bound is $O(K_0(2^n)^{\frac{\alpha}{2} + \frac{1}{\kappa}})$ as before. Otherwise, by a similar reasoning as in (4.28), *the events* $(|X^{(m)}(t-s, y) - y| \gg |y|)$ *are extremely unlikely for n large*, hence we get a *polynomial bound*, $|u^{(m,n-1)}(t, y)| \lesssim K_0(1 + |y|)^{\frac{\alpha}{2} + \frac{1}{\kappa}}$.

- (ii) (bound for $\underline{u}^{(m,n)}(t, x)$, $t \leq T_{\min}(0)$) The exponentially small factors in the right-hand side of (4.16) are not needed for the bound. We replace $\underline{u}^{(m,n)}(t, x) = \mathbb{E}[\cdot]$ with $\mathbb{E}[\mathbf{1}_{X^{(m)}(t, x) \in B(0, 2^n)} \cdot] + \sum_{p \geq n} \mathbb{E}[\mathbf{1}_{X^{(m)}(t, x) \in B(0, 2^{p+1}) \setminus B(0, 2^p)} \cdot]$. Since $|u_0(X^{(m)}(t, x))| \leq K_0(1 + |X^{(m)}(t, x)|)^{\frac{\alpha}{2} + \frac{1}{\kappa}}$, the first

and main term is a $O(e^{-c'(2^n)^{2/\kappa'}})$. Subsequent terms are $O(e^{-c'(2^p)^{2/\kappa'}})$, summing up also to $O(e^{-c'(2^n)^{2/\kappa'}})$.

Let us also bound $\underline{u}^{(m,n)}(t-s, y)$ with $y \in \mathbb{R}^d$ (see (4.22)). Reasoning as in (i), we find: $|\underline{u}^{(m,n)}(t-s, y)| = O(e^{-c'(2^n)^{2/\kappa'}})$ if $\langle y \rangle \leq (2C_\kappa)^{-1}2^{n-1}$, otherwise $|\underline{u}^{(m,n)}(t-s, y)| \lesssim K_0(1+|y|)^{\frac{q}{2}+\frac{1}{\kappa}}$.

(iii) (bound for $w_1^{(m,n)}(t, x)$, $t \leq T_n$) First we use the matrix bound

$$\|T\left(e^{-\int_0^t ds \nabla u^{(m-1)}(t-s, X^{(m)}(t; s, x))}\right)\| \leq \exp\left(\int_0^t ds |\nabla u^{(m-1)}(t-s, X^{(m)}(t; s, x))|\right).$$

Whenever $|X^{(m)}(t; s, x)| > 2^{n+1}$, $e^{-F_{n'}(X^{(m)}(t; s, x)) + |\nabla u^{(m-1)}(t-s, X^{(m)}(t; s, x))|} \leq 1$, $n' = n, n-1$. On the other hand, if $|X^{(m)}(t; s, x)| \leq 2^{n+1}$, then $|\nabla u^{(m-1)}(t-s, X^{(m)}(t; s, x))| \leq C^2 K_1(1+2^{n+1})^{\alpha+\frac{2}{\kappa}}$. Thus the product of the exponential factors is $\leq \exp O\left(T_n \cdot C^2 K_1(1+2^{n+1})^{\alpha+\frac{2}{\kappa}}\right) \leq e$ for C large enough. Then the product of the characteristic function with $\nabla u_0(X^{(m)}(t, x))$ is bounded by $O(e^{-c'(2^n)^{2/\kappa'}})$ by the same arguments as in (ii).

(iv) (bound for $w_2^{(m,n)}(t, x)$ and $w_3^{(m,n)}(t, x)$, $t \leq T_n$) The time-ordered exponential is compensated as in (iii). Proceeding as in (ii), we see that the main contribution comes from the case $X^{(m)}(t; s, x) \in B(0, 2^n)$. Then $|\nabla F_{n'}(X^{(m)}(t; s, x))| \lesssim C^2 K_1(2^n)^{\alpha+\frac{2}{\kappa}-1}$, while $|u^{(m,n-1)}(t-s, X^{(m)}(t; s, x))| \lesssim K_0(2^n)^{\frac{q}{2}+\frac{1}{\kappa}} \leq K_1^{1/2}(2^n)^{\frac{q}{2}+\frac{1}{\kappa}}$. Taking the product with the characteristic function yields $O(e^{-c'(2^n)^{2/\kappa'}})$.

(v) (bound for $w_4^{(m,n)}(t, x)$, $t \leq T_n$) Replace $w_4^{(m,n)}(t; s, x) = \mathbb{E}[\cdot]$ with $\mathbb{E}\left[\mathbf{1}_{X^{(m)}(t; s, x) \notin B(0, (2C_\kappa)^{-1}2^{n-1})} \cdot\right] + \mathbb{E}\left[\mathbf{1}_{X^{(m)}(t; s, x) \in B(0, (2C_\kappa)^{-1}2^{n-1})} \cdot\right]$. The first term is bounded by $O(e^{-c'(2^n)^{2/\kappa'}})$ as in (ii), since (by hypothesis) $|x| \leq (2C_\kappa)^{-1}2^{n-1}$. Assume on the other hand $X^{(m)}(t; s, x) \in B(0, (2C_\kappa)^{-1}2^{n-1})$; then $|\underline{u}^{(m,n)}(t-s, X^{(m)}(t; s, x))| = O(e^{-c'(2^n)^{\alpha+\frac{2}{\kappa}}})$, as proved in (ii).

Leaving aside the bounds for $u^{(m,n)}$ and $\underline{u}^{(m,n)}$, which shall be used in §4.2.2 below, we have proved:

$$|\nabla \underline{u}^{(m,n)}(t, x)| \lesssim e^{-c'(2^n)^{2/\kappa'}}, \quad (4.29)$$

valid for $t \leq T_n$ and $\langle x \rangle \leq (2C_\kappa)^{-1}2^{n-1}$.

For a given dyadic slice

$$x \in B(0, 2^p) \setminus B(0, 2^{p-1}), \quad p \geq 1, \quad (4.30)$$

one may apply this result for any $n \geq n' := p+1 + \lceil 2 \log_2(2C_\kappa) \rceil$.

We now assume $|x| \geq (2C_\kappa)^{\kappa/(\kappa-1)}$ (so that $(C_\kappa - 1)\langle x \rangle^{1/\kappa} \leq \frac{1}{2}|x|$, see (4.27)), fix $n'' := p-1 - \lceil \log_2(2C_\kappa) \rceil \geq 0$ and write

$$\nabla u^{(m)}(t, x) = \nabla u^{(m, n'')}(t, x) + (\nabla \underline{u}^{(m, n''+1)}(t, x) + \dots + \nabla \underline{u}^{(m, n'-1)}(t, x)) + \sum_{n \geq n'} \nabla \underline{u}^{(m, n)}(t, x) \quad (4.31)$$

as a sum of three contributions, in which $|x|$ is *large* (first term, $\nabla u^{(m, n'')}(t, x)$), *small* (last term, $\sum_{n \geq n'} \nabla \underline{u}^{(m, n)}(t, x)$), or of the same order as 2^n .

Our purpose is to show that: $|\nabla u^{(m,n)}(t, x)|$ ($n = n''$), $|\nabla \underline{u}^{(m,n)}(t, x)|$ ($n = n'' + 1, \dots, n' - 1$) are $\lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}$, while the remaining terms, $\nabla \underline{u}^{(m,n)}(t, x)$, $n \geq n'$ are negligible (see (4.29)). The problem, however, is that, for the time being, we shall be able to prove these only for $t \leq T_n$. Since $T_n \rightarrow_{n \rightarrow \infty} 0$, we cannot say anything about the sum in (4.31) till we extend these bounds to arbitrary time (see next subsection).

Consider now the first term (x large) with $t \leq T_{n''}$. As in (i), the contribution coming from the case $\sup_{0 \leq s \leq t} |X^{(m)}(t; s, x) - x| \geq \frac{2}{3}|x|$ is $O(e^{-c(2^{n''})^{2/\kappa'}})$. In the contrary case, $|X^{(m)}(t; s, x)| \leq 2|x|$ for all $0 \leq s \leq t$, so $|\nabla u_0(X^{(m)}(t; s, x))| \lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}$, while

$$\begin{aligned} \int_0^t ds |\nabla F_{n''}(X^{(m)}(t; s, x)) u^{(m,n'')}(t - s, X^{(m)}(t; s, x))| \\ \lesssim T_{n''} \cdot C^2 K_1(1 + |x|)^{\alpha + \frac{2}{\kappa} - 1} \cdot K_0(1 + |x|)^{\frac{\alpha}{2} + \frac{1}{\kappa}} \lesssim K_0(1 + |x|)^{\frac{\alpha}{2} + \frac{1}{\kappa} - 1}. \end{aligned} \quad (4.32)$$

All together we have found: $|\nabla u^{(m,n'')}(t, x)| \lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}$.

Consider finally the finite number of terms $n = n'' + 1, \dots, n' - 1$ for which $|x| \approx 2^n$. Reasoning as in (i) we may assume that $|X^{(m)}(t, x)|, |X^{(m)}(t; s, x)| \lesssim |x|$ in the above formulas, whence

$$w_1^{(m,n)}(t, x) \lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}; \quad (4.33)$$

$$w_2^{(m,n)}(t; s, x), w_3^{(m,n)}(t; s, x), w_4^{(m,n)}(t; s, x) \lesssim C K_1(1 + |x|)^{\alpha + \frac{2}{\kappa} - 1} \cdot K_0(1 + |x|)^{\frac{\alpha}{2} + \frac{1}{\kappa}} \quad (4.34)$$

and finally,

$$|\nabla \underline{u}^{(m,n)}(t, x)| \lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}} + T_n C K_1(1 + |x|)^{\alpha + \frac{2}{\kappa} - 1} \cdot K_0(1 + |x|)^{\frac{\alpha}{2} + \frac{1}{\kappa}} \lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}. \quad (4.35)$$

Clearly the estimates are the same as for $\nabla u^{(m,n'')}$, so in the sequel we shall group together these two terms and rewrite (4.31) as

$$\nabla u^{(m)}(t, x) = \nabla u^{(m,n'-1)}(t, x) + \sum_{n \geq n'} \nabla \underline{u}^{(m,n)}(t, x) \quad (4.36)$$

Note that all these arguments are easily adapted to the case $|x| < (2C_\kappa)^{\kappa/(\kappa-1)}$ provided C is large enough (take $n'' = 0$).

Let us recapitulate. Summing the three contributions from (4.31), or the two contributions from (4.36), we see that (again, provided C is large enough) our induction hypothesis (4.14) should hold at rank m , except that our gradient bounds should be proven to hold for all $t > 0$; and to start with, if possible, for all t less than some uniform stopping time, $t \leq T_{\min}(0) = (C^3 K_1)^{-1}$. This is precisely what we do in the next paragraph.

4.2.2 Large-time bounds for the gradient

For t away from the time origin, bounds for the gradient rest on Schauder estimates. We use a quantitative form of these proved by us in [16]. Let us quote the result for the sake of the reader. More detailed bounds are proved in [16], Proposition 4.5.

Proposition 4.3 [16] *Let v solve the linear parabolic PDE*

$$(\partial_t - \Delta + a(t, x))u(t, x) = b(t, x) \cdot \nabla u(t, x) + f(t, x) \quad (4.37)$$

on the "parabolic ball" $Q^{(j)} = Q^{(j)}(t_0, x_0) := \{(t, x) \in \mathbb{R} \times \mathbb{R}^d; t_0 - 2^j \leq t \leq t_0, x \in \bar{B}(x_0, 2^{j/2})\}$. If u is bounded, $a \geq 0$,

$$\|f\|_{\gamma, Q^{(j)}} := \sup_{(t, x), (t', x') \in Q^{(j)}} \frac{|f(t, x) - f(t', x')|}{|x - x'|^\gamma + |t - t'|^{\gamma/2}} < \infty \quad (4.38)$$

for some $\gamma \in (0, 1)$, and similarly $\|a\|_{\gamma, Q^{(j)}}, \|b\|_{\gamma, Q^{(j)}} < \infty$, then

$$\sup_{Q^{(j-1)}} |\nabla u| \lesssim 2^{j/2} R_b^{-1} \left(2^{j\gamma/2} \|f\|_{\gamma, Q^{(j)}} + (2^{j\gamma} R_b^{-1} \|b\|_{\gamma, Q^{(j)}}^2 + 2^{j\gamma/2} \|a\|_{\gamma, Q^{(j)}} + 2^{-j}) \sup_{Q^{(j)}} |u| \right), \quad (4.39)$$

$$\sup_{Q^{(j-1)}} |\partial_t u|, \sup_{Q^{(j-1)}} |\nabla^2 u| \lesssim R_b^{-1} \left(2^{j\gamma/2} \|f\|_{\gamma, Q^{(j)}} + (2^{j\gamma} R_b^{-1} \|b\|_{\gamma, Q^{(j)}}^2 + 2^{j\gamma/2} \|a\|_{\gamma, Q^{(j)}} + 2^{-j}) \sup_{Q^{(j)}} |u| \right), \quad (4.40)$$

where $R_b := (1 + 2^{j/2} |b(t_0, x_0)|)^{-1}$.

Fix $\gamma \in (0, 1)$ and $x \in B(0, 2^p) \setminus B(0, 2^{p-1})$, $p \geq 1$ in a given dyadic slice. Define $n' := p + 1 + \lceil 2 \log_2(2C_\kappa) \rceil$ as in (4.31)). Recall we have shown: $|\nabla u^{(m, n'-1)}(t, x)| \lesssim K_1(1 + |x|)^{\alpha + \frac{2}{\kappa}}$ for $t \leq T_{n'-1}$, and $|\nabla \underline{u}^{(m, n)}(t, x)| \lesssim e^{-c'(2^n)^{2/\kappa'}} (n \geq n')$ for $t \leq T_n$.

1. We consider first the initial regime $t \leq T_{\min}(0)$, where bounds (i),(ii) for $u^{(m, n)}, \underline{u}^{(m, n)}$ hold (see §4.2.1). Decomposing u as $u^{(m, n'-1)} + \sum_{n \geq n'} \underline{u}^{(m, n)}$, we apply Proposition 4.3, (i) to $u^{(m, n'-1)}$ on $Q := Q^{(\log_2 T_{n'-1})}(t, x)$, $t \geq T_{n'-1}$ (x large); (ii) to $\underline{u}^{(m, n)}$ on $Q := Q^{(\log_2 T_n)}(t, x)$, $t \geq T_n$ for $n \geq n'$ (x small), with $b := -u^{(m-1)}$, $f \equiv 0$ and (i) $a(t, x) := F_n(x)$, (ii) $a \equiv 0$.

We concentrate on case (i), where $2^j = T_{n'-1} \approx T_p \approx (C^3 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}})^{-1}$. Then $R_b^{-1} = 1 + \sqrt{T_{n'-1}} |u^{(m-1)}(t, x)| \lesssim 1$. By Hölder interpolation,

$$\begin{aligned} \|u^{(m-1)}\|_{\gamma, Q} &\lesssim \left(\sup_Q |u^{(m-1)}| \right)^{1-\gamma} \left(\sup_Q |\nabla u^{(m-1)}| \right)^\gamma \lesssim \left(K_0 \langle x \rangle^{\frac{\alpha}{2} + \frac{1}{\kappa}} \right)^{1-\gamma} \left(C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}} \right)^\gamma \\ &\lesssim C^{2\gamma} K_1^{(1+\gamma)/2} \langle x \rangle^{(\frac{\alpha}{2} + \frac{1}{\kappa})(1+\gamma)} \end{aligned} \quad (4.41)$$

since $K_0 \leq K_1^{1/2}$, and $2^{j\gamma/2} \|a\|_{\gamma, Q} \lesssim 2^{j\gamma/2} \cdot C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa} - \gamma} \lesssim C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}}$, $2^{j\gamma} \|u^{(m-1)}\|_{\gamma, Q}^2 + 2^{j\gamma/2} \|a\|_{\gamma, Q^{(j)}} + 2^{-j} \lesssim C^3 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}}$, $2^{j/2} \sup_Q |u^{(m, n'-1)}| \lesssim C^{-3/2}$, hence Proposition 4.3 yields $|\nabla u^{(m, n'-1)}(t, x)| \lesssim C^{3/2} K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}}$.

Consider now briefly (ii) (x small). Then one still has $R_b^{-1} \lesssim 1$, $\|u^{(m-1)}\|_{\gamma, Q} \lesssim C^{2\gamma} K_1^{(1+\gamma)/2} \langle x \rangle^{(\frac{\alpha}{2} + \frac{1}{\kappa})(1+\gamma)}$, while now $T_{n-1}^{1/2} \sup_Q |\underline{u}^{(m, n)}| = O(e^{-c'(2^n)^{2/\kappa'}})$ is exponentially small.

Summing the two contributions, we see that we have proved what we wanted if C is large enough: $|\nabla u^{(m)}(t, x)| \leq C^2 K_1 (1 + |x|)^{\alpha + \frac{2}{\kappa}}$, for all $t \leq T_{\min}(0)$ this time.

2. Let now $t \geq T_{\min}(0)$. Define

$$\langle x \rangle_t := |x| + \langle Ut \rangle^{\kappa/(\kappa-1)}, \quad T_{\min}(t, x) := \left(C^3 K_1 \langle x \rangle_t^{\alpha + \frac{2}{\kappa}} \right)^{-1}. \quad (4.42)$$

Apply Proposition 4.3 directly to u on $Q := Q^{(\log_2 T_{\min}(t, x))}(t, x)$. Then $R_b^{-1} = 1 + \sqrt{T_{\min}(t, x)} |u^{(m-1)}(t, x)| \lesssim 1$. Instead of (4.41) one gets: $\|u^{(m-1)}\|_{\gamma, Q} \lesssim C^{2\gamma} K_1^{(1+\gamma)/2} \langle x \rangle_t^{(\frac{\gamma}{2} + \frac{1}{\kappa})(1+\gamma)}$, whence $T_{\min}(t, x)^\gamma \|u^{(m-1)}\|_{\gamma, Q}^2 + T_{\min}(t, x)^{-1} \lesssim C^3 K_1 \langle x \rangle_t^{\alpha + \frac{2}{\kappa}}$. Finally, $T_{\min}(t, x)^{1/2} \sup_Q |u| \lesssim C^{-3/2}$. Hence Proposition 4.3 yields for C large enough: $|\nabla u^{(m, n'')}(t, x)| \lesssim C^{3/2} K_1 \langle x \rangle_t^{\alpha + \frac{2}{\kappa}}$.

4.2.3 Bounds for $\nabla^2 u^{(m)}$

Unfortunately, in order to prove the convergence of the scheme, we also need to prove bounds for *second-order derivatives* of $u^{(m)}$. However, the proof proceeds exactly as for the gradient, and we shall only sketch it very roughly. We want to prove (1.17) :

(Induction hypothesis)

$$|\nabla^2 u^{(m-1)}(t, x)| \leq C^4 K_2 (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{3(\frac{\gamma}{2} + \frac{1}{\kappa})}. \quad (4.43)$$

Comparing with (4.14), we see that $|\nabla^2 u^{(m-1)}|$ scales roughly like $|\nabla u^{(m-1)}|^{3/2}$. This is coherent with the hypothesis $K_2 \geq K_1^{3/2}$. Differentiating once more the equation for $u^{(m, n)}$ (see Definition 4.2), we get

$$\begin{aligned} (\partial_t - \Delta + (2\nabla u^{(m-1)}(t, x) + F_n(x)) \cdot \nabla) \nabla^2 u^{(m, n)}(t, x) = & -\nabla(\nabla F_n(x) u^{(m, n)}(t, x)) \\ & -\nabla(\nabla u^{(m-1)}(t, x) + F_n(x)) \nabla u^{(m, n)}(t, x). \end{aligned} \quad (4.44)$$

The Feynman-Kac representation for $\nabla^2 u^{(m, n'-1)}$ or $\nabla^2 \underline{u}^{(m, n)}$, $n \geq n'$ is very much alike that of $\nabla u^{(m, n)}$ or $\nabla \underline{u}^{(m, n)}$, except that there is one more gradient, and there appear supplementary terms due to the last term in (4.44). The exponential multiplicative factor is (up to the coefficient 2 in (4.44)) the same as in the case of ∇u , hence may be essentially neglected for $t < T_n$. Similarly, the convection term may be essentially neglected since $|X^{(m-1)}(t, x)| \lesssim \langle x \rangle$ with high probability when $t \leq T_{\min}(0)$. Thus (considering only the main contribution), for $t \lesssim T_{\min}(x) \approx (C^3 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}})^{-1}$, and $n = n' - 1 = \log_2 \langle x \rangle + O(1)$,

$$\begin{aligned} |\nabla^2 u^{(m, n)}(t, x)| \lesssim & \sup_{\langle x' \rangle \approx \langle x \rangle} |\nabla^2 u_0(x')| + T_{\min}(x) \cdot \sup_{0 \leq t' \leq t, \langle x' \rangle \approx \langle x \rangle} \left\{ |\nabla(\nabla F_n(x') u^{(m, n)}(t', x'))| \right. \\ & \left. + |\nabla(\nabla u^{(m-1)}(t', x') + F_n(x'))| \cdot |\nabla u^{(m, n)}(t', x')| \right\}. \end{aligned} \quad (4.45)$$

In this expression $|u^{(m, n)}(t', x')| \lesssim C K_0 \langle x \rangle^{\frac{\alpha}{2} + \frac{1}{\kappa}}$, $|\nabla u^{(m, n)}(t', x')| \lesssim C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}}$, and (by induction) $|\nabla^2 u^{(m-1)}(t', x')| \lesssim C^4 K_2 \langle x \rangle^{3(\frac{\gamma}{2} + \frac{1}{\kappa})}$. The largest terms are obtained by letting the gradient act on $u^{(m, n)}$ since $|\nabla^2 F_n(x')| \lesssim |\nabla F_n(x')| \lesssim F_n(x') \lesssim C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}}$, while bounds on $u^{(m, n)}$ get worse and worse each time one applies a gradient. Hence:

$$\begin{aligned} |\nabla^2 u^{(m, n)}(t, x)| & \lesssim K_2 \langle x \rangle^{3(\frac{\gamma}{2} + \frac{1}{\kappa})} + (C^3 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}})^{-1} \left\{ \left(C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}} \right)^2 + C^4 K_2 \langle x \rangle^{3(\frac{\gamma}{2} + \frac{1}{\kappa})} \cdot C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}} \right\} \\ & \lesssim C^3 K_2 \langle x \rangle^{3(\frac{\gamma}{2} + \frac{1}{\kappa})}. \end{aligned} \quad (4.46)$$

Taking C large enough one obtains inductively a uniform in m short-time estimate for $\nabla^2 u^{(m,n)}$. For t larger one must use Schauder estimates as in §4.2.2 (see (4.42)). Comparing (4.39) with (4.40) one sees that the bound for $\sup_Q |\nabla^2 u^{(m,n)}|$ or $\sup_Q |\nabla^2 u^{(m)}|$ differs from the bound for $\sup_Q |\nabla u^{(m,n)}|$, resp. $\sup_Q |\nabla u^{(m)}|$ only by a multiplicative factor $2^{-j/2} \approx T_p^{-1/2} \approx T_{\min}(t, x)^{-1/2} \approx C^{3/2} K_1^{1/2} \langle x \rangle_t^{\frac{\alpha}{2} + \frac{1}{\kappa}} \leq C^{3/2} \frac{K_2}{K_1} \langle x \rangle_t^{\frac{\alpha}{2} + \frac{1}{\kappa}}$. Hence $\sup_Q |\nabla^2 u^{(m)}| \lesssim C^3 K_2 \langle x \rangle_t^{3(\frac{\alpha}{2} + \frac{1}{\kappa})}$, allowing a bound uniform in m by induction.

4.3 Bounds for $v^{(m)}$

We prove in this section (1.18). Subtracting eq. (1.3) for $m, m-1$, we find an equation for $v^{(m)} := u^{(m)} - u^{(m-1)}$,

$$(\partial_t - \Delta + u^{(m-1)}(t, x) \cdot \nabla) v^{(m)}(t, x) = f^{(m-1)}(t, x) := -v^{(m-1)}(t, x) \cdot \nabla u^{(m-1)}(t, x). \quad (4.47)$$

Recall $T_{\min}(t, x) = \left(C^3 K_1 \langle x \rangle_t^{\alpha + \frac{2}{\kappa}} \right)^{-1}$ (see (4.42)). We assume

(Induction hypothesis)

$$|v^{(m-1)}(t, x)| \leq C K_0 (t/(m-1) T_{\min}(t, x))^{m-1} (|x| + \langle U t \rangle^{\kappa/(\kappa-1)})^{\frac{\alpha}{2} + \frac{1}{\kappa}}, \quad t > 0. \quad (4.48)$$

Note that (4.48) is an improvement on (4.4) only when $t \leq (m-1) T_{\min}(t, x)$, i.e. in some initial regime $t \in [0, T_{\min}^{(m)}(x)]$, where $T_{\min}^{(m)}(x)$ is given by an implicit equation (it is easy to show that $T_{\min}^{(m)}(x) \approx (m-1) T_{\min}(x) \approx (m-1) (C^3 K_1 \langle x \rangle^{\alpha + \frac{2}{\kappa}})^{-1}$ for $\langle x \rangle \geq (U t)^{\kappa/(\kappa-1)}$, in particular for $t \leq U^{-1}$, otherwise $T_{\min}^{(m)}(x) \approx U^{-\lambda} \left(\frac{m-1}{C^3 K_1} \right)^\mu$, with $\lambda = \frac{\kappa\alpha+2}{\kappa(1+\alpha)+1}$, $\mu = \frac{\kappa-1}{\kappa(1+\alpha)+1} < 1$).

Eq. (4.47) also admits a Feynman-Kac representation,

$$v^{(m)}(t, x) = - \int_0^t ds \mathbb{E} \left[v^{(m-1)}(t-s, X^{(m-1)}(t; s, x)) \cdot \nabla u^{(m-1)}(t-s, X^{(m-1)}(t; s, x)) \right]. \quad (4.49)$$

Using the gradient bound, $|\nabla u^{(m-1)}(t, x)| \lesssim C^2 K_1 (|x| + \langle U t \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}}$ and the characteristic estimate $|X^{(m-1)}(t; s, x)| \lesssim |x| + M_t \sqrt{t} + \langle U t \rangle^{\kappa/(\kappa-1)} + \mathbf{1}_{M_t \sqrt{t} \geq \langle U t \rangle^{1/\kappa}} (M_t \sqrt{t})^\kappa$, we deduce (compare with the proof of (3.44)):

$$\begin{aligned} |v^{(m)}(t, x)| &\lesssim \int_0^t ds C K_0 ((t-s)/(m-1) T_{\min}(t, x))^{m-1} \cdot C^2 K_1 (|x| + \langle U t \rangle^{\kappa/(\kappa-1)})^{3(\frac{\alpha}{2} + \frac{1}{\kappa})} \\ &\lesssim K_0 (t/m T_{\min}(t, x))^m (|x| + \langle U t \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}} \\ &\leq C K_0 (t/m T_{\min}(t, x))^m (|x| + \langle U t \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}} \end{aligned} \quad (4.50)$$

for C large enough.

4.4 Gradient bounds for $v^{(m)}$

We prove in this section the bound (1.18) for $\nabla v^{(m)}$.

Differentiating (4.47), one finds

$$(\partial_t - \Delta + u^{(m-1)}(t, x) \cdot \nabla + \nabla u^{(m-1)}(t, x)) \nabla v^{(m)}(t, x) = \nabla f^{(m-1)}(t, x), \quad (4.51)$$

compare with (4.5), with a right-hand side

$$\nabla f^{(m-1)}(t, x) = \nabla \left(-v^{(m-1)}(t, x) \cdot \nabla u^{(m-1)}(t, x) \right) = -v^{(m-1)}(t, x) \cdot \nabla^2 u^{(m-1)}(t, x) - \nabla v^{(m-1)}(t, x) \cdot \nabla u^{(m-1)}(t, x). \quad (4.52)$$

We now proceed as in §4.2 to which we refer for the scheme of proof and notations, and define, similarly to Definition 4.2,

Definition 4.4 (i) For $n \in \mathbb{N}$, let $v^{(m,n)} : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the solution of the transport equation $(\partial_t - \Delta + u^{(m-1)}(t, x) \cdot \nabla) v^{(m,n)}(t, x) = -F_n(x) v^{(m,n)}(t, x) + f^{(m-1)}(t, x)$ with initial condition $v^{(m,n)}(0) = 0$.

(ii) Let $\underline{v}^{(m,n)} := v^{(m,n)} - v^{(m,n-1)}$ ($n \geq 1$).

Differentiating the equation for $v^{(m,n)}$, we get

$$(\partial_t - \Delta + u^{(m-1)}(t, x) \cdot \nabla + (\nabla u^{(m-1)}(t, x) + F_n(x))) \nabla v^{(m,n)}(t, x) = -\nabla F_n(x) v^{(m,n)}(t, x) + \nabla f^{(m-1)}(t, x). \quad (4.53)$$

The Feynman-Kac representation of $v^{(m,n)}$, $\nabla v^{(m,n)}$, $\underline{v}^{(m,n)}$, $\nabla \underline{v}^{(m,n)}$ are totally similar to those of $u^{(m,n)}$, $\nabla u^{(m,n)}$, $\underline{u}^{(m,n)}$, $\nabla \underline{u}^{(m,n)}$, with $u^{(m,n-1)}$, $\underline{u}^{(m,n)}$ replaced by their counterparts $v^{(m,n-1)}$, $\underline{v}^{(m,n)}$ in the expressions for $w_j^{(m,n)}$, $j = 2, 3, 4$, and the initial condition term $w_1^{(m,n)}(t, x)$ replaced by a contribution due to the right-hand side, $\int_0^t ds w_1^{(m,n)}(t, s, x)$, where

$$w_1^{(m,n)}(t, s, x) := \mathbb{E} \left[\mathbf{1}_{X^{(m)}(t; \leq s, x) \notin B(0, 2^{n-1})} \left(e^{-\int_0^s ds' F_n(X^{(m)}(t; s', x))} - e^{-\int_0^s ds' F_{n-1}(X^{(m)}(t; s', x))} \right) T \left(e^{-\int_0^s ds' \nabla u^{(m-1)}(t-s', X^{(m)}(t; s', x))} \nabla f^{(m-1)}(t-s, X^{(m)}(t; s, x)) \right) \right]. \quad (4.54)$$

Fix some exponent $\gamma \in (0, 1)$, and let $\tilde{T}_{min}(t, x) := \left(C^3 K_2^{2/3} (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}} \right)^{-1}$. We assume inductively:

(Induction hypothesis)

$$|\nabla v^{(m-1)}(t, x)| \leq C^3 K_2^{2/3} (t/(m-1) \tilde{T}_{min}(t, x))^{\gamma(m-1)/2} (|x| + \langle Ut \rangle^{\kappa/(\kappa-1)})^{\alpha + \frac{2}{\kappa}}, \quad t \leq (m-1) \tilde{T}_{min}(t, x) \quad (4.55)$$

Let us make two comments at this point. First, because $\nabla f^{(m-1)}(t, x)$ involves the second derivative $\nabla^2 u^{(m-1)}$, which is roughly of order K_2 (for t, x small), and $K_2^{2/3} \geq K_1$, our bounds are in terms of the larger constant $K_2^{2/3}$ and not in terms of K_1 , which also accounts for the replacement of T_{min} by $\tilde{T}_{min} \leq T_{min}$. Second, our bound for $|\nabla v^{(m-1)}(t, \cdot)|$ is in $(t/(m-1))^{\gamma(m-1)/2}$, $\gamma < 1$ instead of the naively expected and smaller $(t/(m-1))^{m-1}$ (as found before for $|v^{(m-1)}(t, \cdot)|$) for reasons that appear only when applying Schauder estimates (see below).

For t small enough, bounds for $\nabla v^{(m,n)}$, $\nabla \underline{v}^{(m,n)}$ may be proved using the Feynman-Kac representation. Let $x \in B(0, 2^p) \setminus B(0, 2^{p-1})$ ($p \geq 1$) and $n' := p + 1 + \lceil 2 \log_2(2C_\kappa) \rceil$ as in (4.31). We refer to

the computations in §4.2.3. Considering only the main contribution, (4.45) is replaced with

$$\begin{aligned}
|\nabla v^{(m,n)}(t, x)| &\lesssim \int_0^t dt' \sup_{\langle x' \rangle \approx \langle x \rangle} (|\nabla F_n(x')| |v^{(m,n)}(t', x')| + |\nabla f^{(m-1)}(t', x')|) \\
&\lesssim \int_0^t dt' \left\{ C^2 K_2^{2/3} \langle x \rangle^{\alpha + \frac{2}{k}} \cdot C K_0(t'/(m-1) \tilde{T}_{\min}(x))^{m-1} \langle x \rangle^{\frac{q}{2} + \frac{1}{k}} \right. \\
&\quad + C K_0(t'/(m-1) \tilde{T}_{\min}(x))^{m-1} \langle x \rangle^{\frac{q}{2} + \frac{1}{k}} \cdot C^4 K_2 \langle x \rangle^{3(\frac{q}{2} + \frac{1}{k})} \\
&\quad \left. + C^3 K_2^{2/3} (t'/(m-1) \tilde{T}_{\min}(x))^{\gamma(m-1)/2} \langle x \rangle^{\alpha + \frac{2}{k}} \cdot C^2 K_1 \langle x \rangle^{\alpha + \frac{2}{k}} \right\} \\
&\lesssim C^3 K_2^{2/3} (t/m \tilde{T}_{\min}(x))^{\gamma m/2} \langle x \rangle^{\alpha + \frac{2}{k}} \tag{4.56}
\end{aligned}$$

for $t \leq m \tilde{T}_{\min}(x)$, where $\tilde{T}_{\min}(x) := \tilde{T}_{\min}(0, x) = (C^3 K_2^{2/3} (1 + |x|)^{\alpha + \frac{2}{k}})^{-1}$.

For larger t , we apply Schauder estimates to eq. (4.47) defining $v^{(m)}$. Compared to §4.2.2, the replacement of $\sup_{Q^{(j)}} |u^{(m)}|$ by $\sup_{Q^{(j)}} |v^{(m)}|$ leads to an extra prefactor $(t/m \tilde{T}_{\min}(t, x))^m \leq (t/m \tilde{T}_{\min}(t, x))^{\gamma m/2}$. However, due to the right-hand side $f^{(m-1)}(t, x) = -v^{(m-1)}(t, x) \cdot \nabla u^{(m-1)}(t, x)$, there appears an extra contribution in the bound (4.39) for $|\nabla v^{(m)}(t, x)|$, namely (concentrating as in §4.2.2 on the main term in the decomposition, for which $2^j \approx \tilde{T}_{\min}(t, x)$),

$$2^{j/2} R_b^{-1} \cdot 2^{j\gamma/2} \|f^{(m-1)}\|_{\gamma, Q^{(j)}} \approx \tilde{T}_{\min}(t, x)^{(1+\gamma)/2} \|v^{(m-1)} \cdot \nabla u^{(m-1)}\|_{\gamma, Q^{(j)}}. \tag{4.57}$$

By induction hypothesis and Hölder interpolation,

$$\begin{aligned}
\|v^{(m-1)} \cdot \nabla u^{(m-1)}\|_{\gamma, Q^{(j)}} &\lesssim \|v^{(m-1)}\|_{\gamma, Q^{(j)}} \|\nabla u^{(m-1)}\|_{\infty, Q^{(j)}} + \|v^{(m-1)}\|_{\infty, Q^{(j)}} \|\nabla u^{(m-1)}\|_{\gamma, Q^{(j)}} \\
&\lesssim \left(\|v^{(m-1)}\|_{\infty, Q^{(j)}}^{1-\gamma} \|\nabla v^{(m-1)}\|_{\infty, Q^{(j)}}^{\gamma} \right) \|\nabla u^{(m-1)}\|_{\infty, Q^{(j)}} + \|v^{(m-1)}\|_{\infty, Q^{(j)}} \left(\|\nabla u^{(m-1)}\|_{\infty, Q^{(j)}}^{1-\gamma} \|\nabla^2 u^{(m-1)}\|_{\infty, Q^{(j)}}^{\gamma} \right) \\
&\lesssim (t/(m-1) \tilde{T}_{\min}(t, x))^{\gamma(m-1)/2} \langle x \rangle_t^{(3+\gamma)(\frac{q}{2} + \frac{1}{k})} \left[(C K_0)^{1-\gamma} (C^3 K_2^{2/3})^{\gamma} C^2 K_1 + C K_0 (C^2 K_1)^{1-\gamma} (C^4 K_2)^{\gamma} \right] \\
&\leq 2C^{3+2\gamma} (K_2^{2/3})^{1+(1+\gamma)/2} (t/(m-1) \tilde{T}_{\min}(t, x))^{\gamma(m-1)/2} \langle x \rangle_t^{(3+\gamma)(\frac{q}{2} + \frac{1}{k})} \\
&\ll C^{3+3(1+\gamma)/2} \left(\langle x \rangle_t^{\alpha + \frac{2}{k}} \right)^{1+(1+\gamma)/2} (K_2^{2/3})^{1+(1+\gamma)/2} (t/(m-1) \tilde{T}_{\min}(t, x))^{\gamma(m-1)/2} \tag{4.58}
\end{aligned}$$

for C large. Upon multiplication by $\tilde{T}_{\min}(t, x)^{(1+\gamma)/2}$ we obtain $2^{j/2} R_b^{-1} \cdot 2^{j\gamma/2} \|f^{(m-1)}\|_{\gamma, Q^{(j)}} \lesssim C^3 K_2^{2/3} (t/(m-1) \tilde{T}_{\min}(t, x))^{\gamma(m-1)/2} \langle x \rangle_t^{\alpha + \frac{2}{k}}$, which is the expected bound for $|\nabla v^{(m)}(t, x)|$, *except* that we still have a factor $(t/(m-1) \tilde{T}_{\min}(t, x))^{\gamma(m-1)/2}$ instead of the required $(t/m \tilde{T}_{\min}(t, x))^{\gamma m/2}$.

By a minor modification of Proposition 4.3, consisting by and large in substituting $\int_{t'}^t ds \|f(s)\|_{\infty, Q^{(j)}(s)}$, $t' < t$ to $(t-t')\|f\|_{\infty, Q^{(j)}}$ where $Q^{(j)}(s)$ is the intersection of the ball $Q^{(j)}$ with the time-slice $t = s$, in order to take advantage of the extra factor in $O(1/m)$ coming from the time integral for $s \ll t$, we are able to extract an extra factor $(t/m \tilde{T}_{\min}(t, x))^{\gamma/2}$ for $t \leq m \tilde{T}_{\min}(t, x)$, where γ is the Hölder exponent. This explains at last why we only obtain a prefactor in $O((t/m \tilde{T}_{\min}(t, x))^{\gamma m/2})$ in the end for the bound (4.55). We do not provide details of this computation since it may be found in our previous article [16], see point (ii) in the proof of Theorem 3.2.

5 Appendix

Lemma 5.1 *Let A_n , $n \geq 0$ be a sequence in \mathbb{R}_+^* satisfying an inductive inequality of the form $A_{n+1} \leq c_1 + c_2 A_n^{\alpha}$, with $c_1, c_2 > 0$ and $\alpha \in (0, 1)$. Then there exists a constant $C_{\alpha} > 0$ depending only on α*

such that $A_n \leq \max(A_0, C_\alpha \max(c_1, c_2^{1/(1-\alpha)}))$ for every $n \geq 1$.

Proof. Clearly $A_n \leq B_n$, where the sequence $(B_n)_{n \geq 0}$ is defined by the inductive relation $B_{n+1} = c_1 + c_2 B_n^\alpha$, with $B_0 = A_0$. Let B^* be the unique positive fixed point of $\phi : B \mapsto c_1 + c_2 B^\alpha$. By standard arguments, $(B_n)_{n \geq 1}$ is increasing (resp. decreasing) if $B_1 \leq B^*$, resp. $B_1 \geq B^*$, and $B_n \rightarrow B^*$. The function $B \mapsto \psi(B) := B - \phi(B)$ ($B \geq 0$) is minimal on $B_* := (\alpha c_2)^{1/(1-\alpha)} \leq c_2^{1/(1-\alpha)}$, and increases on the interval $[B_*, +\infty)$. By construction $B^* \geq B_*$ and $\psi(B^*) = 0$. Let $B_0 := C_\alpha \max(c_1, c_2^{1/(1-\alpha)})$. By definition $B_0 \in [B_*, +\infty)$. We show that $\psi(B_0) \geq 0$, implying $B^* \leq B_0$. There are two cases. If $c_1 \geq c_2^{1/(1-\alpha)}$, then $\psi(B_0) \geq (C_\alpha - 1 - C_\alpha^\alpha) c_1$. In the contrary case, $\psi(B_0) \geq (C_\alpha - 1 - C_\alpha^\alpha) c_2^{1/(1-\alpha)}$. Thus in both cases $\psi(B_0) \geq 0$ provided C_α is large enough. \square

References

- [1] J. Bec, K. Khanin. *Burgers turbulence*, Phys. Rep. **447**, 1–66 (2007).
- [2] A. J. Majda, A. L. Bertozzi. *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press (2002).
- [3] F. Da Lio, O. Ley. *Uniqueness results for second order Bellman-Isaacs equations under quadratic growth assumptions and applications*, SIAM J. Control Optim. 45 (2006), no. 1, 74–106.
- [4] G. Da Prato, A. Debussche, R. Temam. *Stochastic Burgers equation*, NoDEA Nonlinear Differential Equations Appl. **1** (4), 389–402 (1994).
- [5] A. Dermoune. *Around the stochastic Burgers equation*, Stoch. Anal. Appl. **15** (3), 295–311 (1997).
- [6] Weinan E, K. Khanin, A. Mazel, Y. Sinai. *Invariant measures for Burgers equation with stochastic forcing*, Annals of Mathematics **151**, 877–960 (2000).
- [7] A. Friedman. *Partial differential equations of parabolic type*, Prentice-Hall (1964).
- [8] C. Kipnis, C. Landim. *Scaling limits of interacting particle systems*, Grundlehren der Mathematischen Wissenschaften, vol. 320, Springer-Verlag (1999).
- [9] A. A. Kiselev, O. A. Ladyzhenskaja. *On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid* (in Russian), Izv. Akad. Nauk SSSR. Ser. Mat. 21, 655–680 (1957).
- [10] G. M. Lieberman. *Second order parabolic differential equations*, World Scientific (1996).
- [11] D. Revuz, M. Yor. *Continuous martingales and Brownian motion*, Grundlehren der Mathematischen Wissenschaften **293**. Springer-Verlag, Berlin (1999).
- [12] H. Spohn. *Large scale dynamics of interacting particles*, Texts and Monographs in Physics, Springer-Verlag (1991).
- [13] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals*, Princeton University Press (1993).

- [14] M. Taylor. *Partial differential equations. III Nonlinear equations*, Applied Mathematical Sciences, vol. 117, Springer (2011).
- [15] H. Triebel. *Theory of function spaces. II*, Monographs in Mathematics **84**. Birkhäuser, Basel (1992).
- [16] J. Unterberger. *Global existence for strong solutions of viscous Burgers equation. (1) The bounded case*, [arXiv.org/abs/1503.05145](https://arxiv.org/abs/1503.05145).
- [17] X.-J. Wang. *Schauder estimates for elliptic and parabolic equations*, Chin. Ann. Math. **27B(6)**, 637-642 (2006).